Egorychev method and the evaluation of combinatorial sums: Part 3: complex variables

Marko R. Riedel

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The Egorychev method is from the book by G.P.Egorychev [Ego84]. We collect several examples, the focus being on computational methods to produce results. Those that are from posts to math.stackexchange.com have retained the question answer format from that site. The website for this document is at this hyperlink:
https://pnp.mathematik.uni-stuttgart.de/iadm/Riedel/egorychev.html.

The crux of the method is the use of integrals from the Cauchy Residue Theorem to represent binomial coefficients, exponentials, the Iverson bracket and Stirling numbers, Catalan numbers, Harmonic numbers, Eulerian numbers and Bernoulli numbers. There is a tutorial at the following article: [RM23].

We use these types of integrals:

- *First binomial coefficient integral (B_1)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{k+1}}\; dz = \operatorname*{res}_z \frac{(1+z)^n}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Second binomial coefficient integral (B_2)*

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{(1-z)^{k+1} z^{n-k+1}}\; dz = \operatorname*{res}_z \frac{1}{(1-z)^{k+1} z^{n-k+1}}$$

  where $0 < \varepsilon < 1$.

- *Exponentiation integral (E)*

$$n^k = \frac{k!}{2\pi i}\int_{|z|=\varepsilon} \frac{\exp(nz)}{z^{k+1}}\; dz = k! \operatorname*{res}_z \frac{\exp(nz)}{z^{k+1}}$$

  where $0 < \varepsilon < \infty$.

- *Iverson bracket (I)*

$$[[k \leq n]] = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz = \operatorname*{res}_{z} \frac{z^k}{z^{n+1}} \frac{1}{1-z}$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the first kind*

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_{z} \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k$$

  where $0 < \varepsilon < 1$.

- *Stirling numbers of the second kind*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \left(\exp(z) - 1 \right)^k \; dz = \frac{n!}{k!} \operatorname*{res}_{z} \frac{1}{z^{n+1}} \left(\exp(z) - 1 \right)^k$$

  where $0 < \varepsilon < \infty$.

The residue at infinity is coded R.

Contents

List of identities in this document

section 1 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{k}{j}=(-1)^n\binom{n}{j}\binom{n+j}{j}.$$

section 2 B_1B_2

$$\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n}=\binom{s+r+1}{n+m+1}.$$

section 3 B_1

$$\sum_{q=0}^{2m}(-1)^q\binom{p-1+q}{q}\binom{2m+2p+q-1}{2m-q}2^q=(-1)^m\binom{p-1+m}{m}.$$

section 4 B_1

$$\sum_{k=0}^{\lfloor m/2\rfloor}\binom{n}{k}(-1)^k\binom{m-2k+n-1}{n-1}=\binom{n}{m}.$$

section 5 B_2

$$\sum_{k=0}^{n}k\binom{2n}{n+k}=\frac{1}{2}n\binom{2n}{n}.$$

section 6 I_1

$$\sum_{k=0}^{n}2^{-k}\binom{n+k}{k}=2^n.$$

section 7 B_1

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n}\frac{1}{a+bk+1}\binom{a+bk}{m}\binom{k-n-1}{k}=\binom{n}{m}.$$

section 8 B_1, R

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n}=\binom{2m}{2n}.$$

section 9 B_1E

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q}\binom{q}{r}(-1)^q q^{p-r} = \frac{p!}{r!}.$$

section 10 B_1

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor} = \binom{2n+1}{n}.$$

section 11 B_1

Verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n} \frac{(2k+2v)!}{(k+v)! \times v! \times (2k+v)! \times (n-v)!} 2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{(k+m)! \times m! \times (n-2m)!} 2^{n-4m}.$$

section 12 B_1

If

$$T(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} T(n-k)$$

for $n \geq 2$ then

$$T(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

section 13 B_1

$$\sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^{k+l} \binom{n+k-l}{n} \binom{k+l}{n} \binom{n}{k} \binom{n}{l} = (-1)^m \binom{2m}{m}.$$

section 14 B_1

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

section 15

$$\sum_{k=m}^{n} (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{n}{k} \binom{n+k}{k}^{-1} \binom{k}{m} \binom{k+m}{m} = \delta_{mn}.$$

section 16 B_1

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}^2\binom{2k}{n}.$$

section 17

$$\sum_{s}\binom{n+s}{k+\ell}\binom{k}{s}\binom{\ell}{s}=\binom{n}{k}\binom{n}{\ell}.$$

section 18 B_1

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k\binom{2n}{n+3k}=2\times 3^{n-1}.$$

section 19 B_1

$$\sum_{k=0}^{\rho}\binom{2x+1}{2k}\binom{x-k}{\rho-k}=\frac{2x+1}{2\rho+1}\binom{x+\rho}{2\rho}2^{2\rho}.$$

section 20 B_1B_2

$$\sum_{k=0}^{\min(a,b)}\binom{x+y+k}{k}\binom{x}{b-k}\binom{y}{a-k}=\binom{x+a}{b}\binom{y+b}{a}$$

section 21 B_1I

$$\sum_{q=0}^{n}\binom{n}{2q}\binom{n-2q}{p-q}2^{2q}=\binom{2n}{2p}.$$

section 22 B_1R

$$\sum_{k=0}^{n-1}\left(\sum_{q=0}^{k}\binom{n}{q}\right)\left(\sum_{q=k+1}^{n}\binom{n}{q}\right)=\frac{1}{2}n\binom{2n}{n}.$$

section 23 B_1

$$(1-x)^{2k+1}\sum_{n\geq 0}\binom{n+k-1}{k}\binom{n+k}{k}x^n=\sum_{j\geq 0}\binom{k-1}{j-1}\binom{k+1}{j}x^j.$$

section 24 B_1

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-3k}{n} = \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n+1}{k} \binom{k}{n-k}.$$

section 25 B_1

$$\sum_{k=0}^{\lfloor (m+n)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1} = \binom{n}{m+1}.$$

section 26 B_1

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} F_{k+1} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} F_{2k+1}.$$

section 27 $B_2 R$

$$\sum_{p,q \geq 0} \binom{n-p}{q} \binom{n-q}{p} = F_{2n+2}.$$

section 28 $B_1 I$

$$\sum_{r=0}^{n} \binom{r+n-1}{n-1} \binom{3n-r}{n} = \frac{1}{2} \left(\binom{4n}{2n} + \binom{2n}{n}^2 \right).$$

section 29 B_1

$$\sum_{r=1}^{n+1} \frac{1}{r+1} \binom{2r}{r} \binom{m+n-2r}{n+1-r} = \binom{m+n}{n}.$$

section 30 B_1

$$\sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

section 31 B_1

$$\sum_{k} \binom{tk+r}{k} \binom{tn-tk+s}{n-k} \frac{r}{tk+r} = \binom{tn+r+s}{n}$$

section 32 B_1

$$\sum_{q=0}^{n-2} \sum_{k=1}^{n} \binom{k+q}{k} \binom{2n-q-k-1}{n-k+1} = n \times \binom{2n}{n+2}$$

section 33 B_1I

$$\sum_{q=0}^{l}\binom{q+k}{k}\binom{l-q}{k}=\binom{l+k+1}{2k+1},$$

section 34 B_1I

$$\sum_{k=0}^{n}k\binom{m+k}{m+1}=\frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

section 35 B_1IR

$$\sum_{k=0}^{n}2^{n-k}\binom{k}{\lfloor k/2\rfloor}=-2^n+(2n+2+(n \mod 2))\binom{n}{\lfloor n/2\rfloor}.$$

section 36 IR

$$\sum_{q=0}^{m-1}\binom{n-1+q}{q}x^n(1-x)^q+\sum_{q=0}^{n-1}\binom{m-1+q}{q}x^q(1-x)^m=1$$

where $n,m\geq 1$

as well as

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k}+\sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}=2^{n+m+1}.$$

section 37 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{p+q+1}{k}\binom{p+n-k}{n-k}\binom{q+n-k}{n-k}=\binom{p}{n}\binom{q}{n}.$$

section 38 B_1

$$\sum_{k=0}^{n}\binom{n}{k}\binom{pn-n}{k}\binom{pn+k}{k}=\binom{pn}{n}^2.$$

section 39 B_1

$$\sum_{r=0}^{\min\{m,n,p\}}\binom{m}{r}\binom{n}{r}\binom{p+m+n-r}{m+n}=\binom{p+m}{m}\binom{p+n}{n}.$$

section 40 B_1

$$\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^q\binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q}=2^l\binom{m-n}{l}$$

section 41 E

$$\sum_{q=0}^{n}(n-2q)^k\binom{n}{2q+1}=\sum_{q=0}^{k+1}\binom{n}{q}2^{n-q-1}\times q!\times\left\{{k+1 \atop q+1}\right\}-\frac{1}{2}\times n!\times\left\{{k+1 \atop n+1}\right\}.$$

section 42 B_1B_2R

$$\sum_{q=0}^{n}q\binom{2n}{n+q}\binom{m+q-1}{2m-1}=m\times 4^{n-m}\times\binom{n}{m}$$

where $n\geq m$.

section 43 B_1B_2R

$$\sum_{q=0}^{n}q\binom{2n}{n+q}\binom{m+q-1}{2m-1}=m\times 4^{n-m}\times\binom{n}{m}$$

where $n\geq m$.

(different proof).

section 44 B_1EIR

With

$$b_k^n=\sum_{l=1}^{k}(-1)^{k-l}l^n\binom{n+1}{k-l}$$

and we

show that $b_k^n=b_{n+1-k}^n$ where $0\leq k\leq n+1$.

section 45 B_1B_2

Suppose we have a random variable X where

$$\mathrm{P}[X=k]=\binom{N}{2n+1}^{-1}\binom{N-k}{n}\binom{k-1}{n}$$

for $k=n+1,\ldots,N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

section 46 B_1

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

section 47 B_1

Same as previous, generalized.

section 48 B_1

$$(-1)^m \frac{(n+m)!}{(n-m)!}\left(\frac{d}{dz}\right)^{n-m}(1-z^2)^n = (1-z^2)^m\left(\frac{d}{dz}\right)^{n+m}(1-z^2)^n$$

section 49 B_1

$$r^k(r+n)! = \sum_{m=0}^{k}(r+n+m)!(-1)^{k+m}\sum_{p=0}^{k-m}\binom{k}{p}\left\{{k+1-p \atop m+1}\right\}n^p.$$

section 50 B_1

$$\sum_{k=0}^{n}\sum_{q=0}^{k}(-1)^q\binom{k}{q}\binom{n-1-qm}{k-1} = [z^n]\frac{1}{1-w-w^2-\cdots-w^m}.$$

section 51

$$\left\{{n \atop m}\right\} = \sum_{k=m}^{n}\binom{k}{m}\sum_{q=0}^{k}(-1)^n\left\{{n+q-m \atop k}\right\}(-1)^k\left[{k \atop q}\right]\binom{n}{n+q-m}$$

section 52 B_1

$$\sum_{k=0}^{m}\frac{q}{pk+q}\binom{pk+q}{k}\binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

section 53 B_1B_2

$$\sum_{k=q}^{n-1}\frac{q}{k}\binom{2n-2k-2}{n-k-1}\binom{2k-q-1}{k-1}=\binom{2n-q-2}{n-1}.$$

section 54

$$\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}\binom{n+m}{n-m}=(-1)^n2^{2n}.$$

section 55

$$\sum_{\substack{k=0\\k\text{ odd}}}^{m}\binom{2n}{2n-k}\binom{2m-2n}{m-k}=\frac{1}{2}\binom{2m}{m}+(-1)^{m+1}2^{2m-1}\binom{n-1/2}{m}$$

section 56

$$\sum_{j=0}^{n}(-1)^{n+j}\left[{n \atop j}\right]\left\{{m+j \atop k}\right\}=\frac{n!}{k!}\sum_{q=0}^{k}\binom{k}{q}\binom{q}{n}(-1)^{k-q}q^m.$$

section 57

$$[z^k]\frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n=\binom{n+2k}{k}$$

section 58.1

$$\sum_{k=1}^{m-1}\sin^{2q}(k\pi/m)=m\frac{1}{2^{2q}}\binom{2q}{q}+m\frac{1}{2^{2q-1}}\sum_{l=1}^{\lfloor q/m\rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

section 58.2 B_1

$$\begin{aligned}&\sum_{j=-\lfloor n/p\rfloor}^{\lfloor n/p\rfloor}(-1)^j\binom{2n}{n-pj}\\&\quad=[z^n]\left(\sum_{q=0}^{\lfloor p/2\rfloor}\frac{p}{p-q}\binom{p-q}{q}(-1)^qz^q\right)^{-1}\sum_{q=0}^{\lfloor (p-1)/2\rfloor}\binom{p-1-q}{q}(-1)^qz^q\end{aligned}$$

section 58.3 B_1

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k}=\sum_{k=0}^{n}\binom{m+1}{k}.$$

section 58.4

$$\sum_{m\geq 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=0}^{n}\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k$$

section 58.5

$$\sum_{n\geq 0} \left\{ {n+r \atop n} \right\} z^n = \frac{1}{(1-z)^{2r+1}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle z^k$$

section 58.5.1

$$\sum_{n\geq 0} \left[{n+r+1 \atop n+1} \right] z^n = \frac{1}{(1-z)^{2r+1}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^k$$

section 58.6

$$\binom{q-j+k}{k} + (-1)^k\binom{j}{k} = \sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{q/2+\ell}{2\ell}\left(\binom{q/2-j+k-\ell}{k-2\ell} + \binom{q/2-j+k-\ell-1}{k-2\ell}\right)$$

section 58.7

$$\sum_{j=0}^{k}\binom{2j}{j+q}\binom{2k-2j}{k-j} = 4^k - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}$$

section 58.8

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n} = \binom{2m}{2n}$$

section 58.9

$$\sum_{q=0}^{n}\binom{q}{n-q}(-1)^{n-q}\binom{2q+1}{q+1} = 2^{n+1}-1$$

section 58.10

$$\sum_{k=0}^{n}\left\{ {n+k \atop k} \right\}\binom{2n}{n+k}\frac{(-1)^k}{k+1} = B_n\binom{2n}{n}\frac{1}{n+1}$$

section 58.11

With $f(z)$ the OGF and $g(w)$ the EGF of a sequence we have

$$g(w) = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{f(z)}{z}\exp(w/z)\,dz$$

section 58.12

$$\sum_{q=0}^{r}(-1)^{q+r}\begin{bmatrix} r \\ q \end{bmatrix}\left\{ \begin{matrix} n+q-1 \\ k \end{matrix} \right\} = \frac{(-1)^{k-r}}{(k-r)!}\sum_{p=0}^{k-r}\binom{k-r}{p}(-1)^p(p+r)^{n-1}$$

section 58.13

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{m} = \frac{n}{m}\binom{n-m-1}{m-1}2^{n-2m}$$

section 58.14

$$[z^n]\log^2\frac{2}{1+\sqrt{1-4z}} = \binom{2n}{n}(H_{2n-1}-H_n)\frac{1}{n}$$

section 58.15

$$\sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}B_k = \frac{n!}{n+1}$$

section 58.16

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q} = \frac{1}{2}(1+(-1)^K)$$

section 58.17

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\left[\frac{1}{f^k(x)}\right]^{(n)} f^{k+1}(x) = [f(x)]^{(n)}$$

section 58.18

$$2^n\binom{2r}{r}\binom{2r+n}{r}\binom{r+n}{n/2} = \sum_{k=0}^{n}(-1)^k\binom{k+r}{r}\binom{2k+2r}{k+r}\binom{n-k+r}{r}\binom{2n-2k+2r}{n-k+r}$$

section 58.19

$$\begin{bmatrix} n \\ m \end{bmatrix} = (-1)^{n-m}\frac{(n-1)!}{(m-1)!}[z^{n-m}]\left(\frac{z}{\exp(z)-1}\right)^n$$

section ??

$$2^{n-m}3^m = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{m+3k}{k}\binom{n+2k}{n-k}$$

1 Introductory example for the method (B_1)

Suppose we seek to evaluate

$$S_j(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{j}$$

which is claimed to be

$$(-1)^n \binom{n}{j} \binom{n+j}{j}.$$

Introduce

$$\binom{n+k}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{k+1}} \; dz$$

and

$$\binom{k}{j} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^k}{w^{j+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1+z)^k (1+w)^k}{z^k} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \left(1 - \frac{(1+w)(1+z)}{z}\right)^n \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} (-1-w-wz)^n \; dw \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} (1+w+wz)^n \; dw \; dz.$$

This is

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{q=0}^{n} \binom{n}{q} w^q (1+z)^q \; dw \; dz.$$

Extracting the residue at $w = 0$ we get

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \binom{n}{j} (1+z)^j \; dz$$

$$= \binom{n}{j} \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+j}}{z^{n+1}} \; dz$$

$$= (-1)^n \binom{n}{j} \binom{n+j}{n}.$$

thus proving the claim.

This is math.stackexchange.com problem 1331507.

2 Introductory example for the method, convergence about zero (B_1B_2)

Suppose we seek to evaluate

$$\sum_{k=0}^{r} \binom{r-k}{m}\binom{s+k}{n}$$

where $n \ge s$ and $m \le r$. We then have it for non-negative r and arbitrary s because the sum is a polynomial in r and s and we have agreement on an infinite number of points.

Introduce

$$\binom{r-k}{m} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r-k-m+1}} \frac{1}{(1-z)^{m+1}} \; dz.$$

Note that this is zero when $k > r-m$ and $\epsilon < 1$ so we may extend the sum in k to infinity. Observe here that we get a zero value for $\binom{r-k}{m}$ when $k > r$ i.e. the upper index becomes negative. We do not get $(r-k)^{\underline{m}}/m!$ which does not vanish. Introduce furthermore

$$\binom{s+k}{n} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{s+k}}{w^{n+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \sum_{k\ge 0} z^k (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \frac{1}{1-(1+w)z} \; dw \; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r-m+2}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \frac{1}{w-(1-z)/z} \; dw \; dz$$

For the geometric series to converge we must have $|z(1+w)| < 1$. Observe that $|z(1+w)| = \epsilon|1+w| \le \epsilon(1+\gamma)$. So we need to choose $1+\gamma < 1/\epsilon$ with ϵ in a neighborhood of zero. We also want the pole at $w = (1-z)/z$ to be outside the contour in w. Now $|(1-z)/z| > (1-\epsilon)/\epsilon$ so we need the latter to be larger than γ. Taking $\gamma = \epsilon$ we require from the pole $1-\epsilon > \epsilon^2$ or $\epsilon < (\sqrt{5}-1)/2$. For the convergence we need $1+\epsilon < 1/\epsilon$ or once again $\epsilon < (\sqrt{5}-1)/2$. Therefore the choice $\epsilon = 1/2$ and $\gamma = 1/2$ will work.

Continuing we evaluate the inner integral with minus the residue at $w =$

$(1-z)/z$ because residues sum to zero, the residue at infinity is zero by inspection and the pole is outside the contour. We find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r-m+2}} \frac{1}{(1-z)^{m+1}} \frac{1}{z^s} \frac{z^{n+1}}{(1-z)^{n+1}}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{r+s-n-m+1}} \frac{1}{(1-z)^{n+m+2}}\; dz.$$

This yields

$$\binom{r+s-n-m+n+m+1}{n+m+1} = \binom{r+s+1}{n+m+1}$$

which concludes the argument.

Alternate proof

Start with

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^r}{z^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \sum_{k=0}^{r} \frac{(1+w)^k}{(1+z)^k}\; dw\; dz.$$

where $\varepsilon > \gamma$ and this time we require $n \ge s$ and $n, m \ge 0$. Note that here $\binom{r-k}{m}$ does not vanish when $k > r$ but those k are outside the range of our sum. We obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^r}{z^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \frac{1-(1+w)^{r+1}/(1+z)^{r+1}}{1-(1+w)/(1+z)}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^r}{z^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \frac{1+z-(1+w)^{r+1}/(1+z)^r}{z-w}\; dw\; dz.$$

The sum was finite so no convergence issues. Now we have two pieces, the first piece is

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{r+1}}{z^{m+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^s}{w^{n+1}} \frac{1}{z-w}\; dw\; dz.$$

Note that with our choice of contour the pole at $w = z$ of the integral in w is outside the contour. With $n \ge s$ the residue at infinity is zero so we may evaluate the integral with minus the residue at $w = z$. We get

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{r+s+1}}{z^{m+n+2}}\; dz = \binom{r+s+1}{m+n+1}.$$

This is the claim. The second piece gives

$$-\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{m+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{r+s+1}}{w^{n+1}}\frac{1}{z-w}\;dw\;dz.$$

Here we note that when $n+2-(r+s+1)<2$ the residue at infinity does not vanish, hence we cannot evaluate by the method that we used on the first piece. Instead we apply the Leibniz rule to the integral in w to get

$$\frac{1}{n!}\left((1+w)^{r+s+1}\frac{1}{(z-w)^1}\right)^{(n)}$$
$$=\frac{1}{n!}\sum_{q=0}^{n}\binom{n}{q}(1+w)^{r+s+1-q}(r+s+1)^{\underline{q}}\frac{1}{(z-w)^{1+n-q}}1^{\overline{n-q}}.$$

Evaluate at $w=0$ and simplify geometrics,

$$\sum_{q=0}^{n}\binom{r+s+1}{q}\frac{1}{z^{1+n-q}}.$$

Note however that

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{m+1}}\frac{1}{z^{n+1-q}}\;dz=[[m+n+1-q=0]]=0$$

owing to the boundary conditions, which once more concludes the argument.

Remark. If we insist on evaluating the second piece with the residue at infinity we get from the pole at $w=z$ a value of $-\binom{r+s+1}{n+m+1}$, where we have flipped the sign. We get from minus the residue at infinity

$$\begin{aligned}&\operatorname*{res}_{w}\frac{1}{w^2}w^{n+1}\frac{(1+w)^{r+s+1}}{w^{r+s+1}}\frac{1}{1/w-z}\\ &=\operatorname*{res}_{w}\frac{(1+w)^{r+s+1}}{w^{r+s-n+1}}\frac{1}{1-wz}\\ &=\sum_{q=0}^{r+s-n}\binom{r+s+1}{q}z^{r+s-n-q}\end{aligned}$$

With the remaining integral in z the only non-zero contribution is from $q=r+s-n-m$, giving $\binom{r+s+1}{r+s-n-m}=\binom{r+s-1}{n+m+1}$. Adding the two contributions produces a zero value as before.

This was math.stackexchange.com problem 928271.

3 Introductory example for the method, an interesting substitution (B_1)

Suppose we seek to verify that

$$\sum_{q=0}^{2m} (-1)^q \binom{p-1+q}{q} \binom{2m+2p+q-1}{2m-q} 2^q = (-1)^m \binom{p-1+m}{m}.$$

Introduce

$$\binom{2m+2p+q-1}{2m-q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-q+1}} (1+z)^{2m+2p+q-1} \; dz.$$

Observe that this controls the range being zero when $q > 2m$ so we may extend q to infinity to obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \sum_{q \ge 0} \binom{p-1+q}{q} (-1)^q 2^q z^q (1+z)^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \frac{1}{(1+2z(z+1))^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \frac{1}{((1+z)^2+z^2)^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m-1} \frac{1}{(1+z^2/(1+z)^2)^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m}} (1+z)^{2m} \frac{1}{z(1+z)} \frac{1}{(1+z^2/(1+z)^2)^p} \; dz.$$

Now put

$$\frac{z}{1+z} = u \quad \text{so that} \quad z = \frac{u}{1-u} \quad \text{and} \quad dz = \frac{1}{(1-u)^2} du$$

to obtain for the integral

$$\frac{1}{2\pi i} \int_{|u|=\gamma} \frac{1}{u^{2m}} \frac{1}{u/(1-u) \times 1/(1-u)} \frac{1}{(1+u^2)^p} \frac{1}{(1-u)^2} \; du$$

$$= \frac{1}{2\pi i} \int_{|u|=\gamma} \frac{1}{u^{2m+1}} \frac{1}{(1+u^2)^p} \; du.$$

This is

$$[u^{2m}] \frac{1}{(1+u^2)^p} = [v^m] \frac{1}{(1+v)^p} = (-1)^m \binom{m+p-1}{m},$$

as claimed.

For the conditions on ϵ and γ we require convergence of the geometric series with $|2z(1+z)| < 1$ which holds for $\epsilon < (-1+\sqrt{3})/2$. Note that with $u = z + \cdots$ the image of $|z| = \epsilon$ makes one turn around zero. The closest it comes to the origin is at $\epsilon/(1+\epsilon)$ so we must choose $\gamma < \epsilon/(1+\epsilon)$ e.g. $\gamma = \epsilon^2/(1+\epsilon)$ for $|w| = \gamma$ to be entirely contained in the image of $|z| = \epsilon$. The poles at $u = \pm i$ are definitely outside this image. Taking $\epsilon = 1/5$ will work.

This was math.stackexchange.com problem 557982.

4 Introductory example for the method, another interesting substitution (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k} (-1)^k \binom{m-2k+n-1}{n-1}$$

where $m \le n$ and introduce

$$\binom{m-2k+n-1}{n-1} = \binom{m-2k+n-1}{m-2k}$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m-2k+1}} (1+z)^{m-2k+n-1} \; dz$$

which has the property that it is zero when $2k > m$ so we may set the upper limit in the sum to n, getting

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} (1+z)^{m+n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{z^{2k}}{(1+z)^{2k}} \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} (1+z)^{m+n-1} \left(1 - \frac{z^2}{(1+z)^2}\right)^n \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} (1+z)^{m-n-1} (1+2z)^n \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^m}{z^m} \frac{1}{z(1+z)} \frac{(1+2z)^n}{(1+z)^n} \; dz.$$

Now put

$$\frac{1+2z}{1+z} = u \quad \text{so that} \quad z = -\frac{u-1}{u-2}, \; 1+z = -\frac{1}{u-2}, \; \frac{1+z}{z} = \frac{1}{u-1},$$

$$\frac{1}{z(1+z)} = \frac{(u-2)^2}{u-1} \quad \text{and} \quad dz = \frac{1}{(u-2)^2} \; du$$

to get for the integral

$$\frac{1}{2\pi i}\int_{|u-1|=\gamma} \frac{1}{(u-1)^m}\frac{(u-2)^2}{u-1}u^n\frac{1}{(u-2)^2}\; du$$
$$= \frac{1}{2\pi i}\int_{|u-1|=\gamma} \frac{1}{(u-1)^{m+1}}u^n\; du.$$

This is

$$[(u-1)^m]u^n = [(u-1)^m]\sum_{q=0}^{n}\binom{n}{q}(u-1)^q = \binom{n}{m}.$$

This solution is more complicated than the obvious one (which can be found at the stackexchange link) but it serves to illustrate the substitution aspect of the method.

Concerning the choice of ϵ and γ the closest that the image of $|z|=\epsilon$ which is $1+\frac{z}{1+z}$, gets to one, is $\epsilon/(1+\epsilon)$ so that must be the upper bound for γ. Taking $\epsilon = 1/3$ and $\gamma = 1/5$ will work. Note also that $u = 1 + z + \cdots$ makes one turn around one.

This was math.stackexchange.com problem 1558659.

5 Introductory example for the method, yet another interesting substitution (B_2)

Suppose we seek to evaluate

$$\sum_{k=0}^{n} k\binom{2n}{n+k}.$$

Introduce

$$\binom{2n}{n+k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-k+1}}\frac{1}{(1-z)^{n+k+1}}\; dz.$$

Observe that this is zero when $k > n$ so we may extend k to infinity to obtain for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}\frac{1}{(1-z)^{n+1}}\sum_{k\ge 0} k\frac{z^k}{(1-z)^k}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}\frac{1}{(1-z)^{n+1}}\frac{z/(1-z)}{(1-z/(1-z))^2}\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n}}\frac{1}{(1-z)^{n}}\frac{1}{(1-2z)^2}\; dz.$$

Now put $z(1-z) = w$ so that (observe that with $w = z + \cdots$ the image of $|z| = \epsilon$ with ϵ small is another closed circle-like contour which makes one turn and which we may certainly deform to obtain another circle $|w| = \gamma$)

$$z = \frac{1 - \sqrt{1-4w}}{2} \quad \text{and} \quad (1-2z)^2 = 1 - 4w$$

and furthermore

$$dz = -\frac{1}{2} \times \frac{1}{2} \times (-4) \times (1-4w)^{-1/2} \; dw = (1-4w)^{-1/2} \; dw$$

to get for the integral

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-4w} (1-4w)^{-1/2} \; dw = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{(1-4w)^{3/2}} \; dw.$$

This evaluates by inspection to

$$4^{n-1} \binom{n-1+1/2}{n-1} = 4^{n-1} \binom{n-1/2}{n-1} = \frac{4^{n-1}}{(n-1)!} \prod_{q=0}^{n-2} (n - 1/2 - q)$$

$$= \frac{2^{n-1}}{(n-1)!} \prod_{q=0}^{n-2} (2n - 2q - 1) = \frac{2^{n-1}}{(n-1)!} \frac{(2n-1)!}{2^{n-1}(n-1)!}$$

$$= \frac{n^2}{2n} \binom{2n}{n} = \frac{1}{2} n \binom{2n}{n}.$$

Here the mapping from $z = 0$ to $w = 0$ determines the choice of square root. For the conditions on ϵ and γ we have that for the series to converge we require $|z/(1-z)| < 1$ or $\epsilon/(1-\epsilon) < 1$ or $\epsilon < 1/2$. The closest that the image contour of $|z| = \epsilon$ comes to the origin is $\epsilon - \epsilon^2$ so we choose $\gamma < \epsilon - \epsilon^2$ for example $\gamma = \epsilon^2 - \epsilon^3$. This also ensures that $\gamma < 1/4$ so $|w| = \gamma$ does not intersect the branch cut $[1/4, \infty)$ (and is contained in the image of $|z| = \epsilon$). For example $\epsilon = 1/3$ and $\gamma = 2/27$ will work.

This was math.stackexchange.com problem 1585536.

Using formal power series

We may use the change of variables rule 1.8 (5) from the Egorychev text (page 16) on the integral

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^n} \frac{1}{(1-z)^n} \frac{1}{(1-2z)^2} \; dz = \operatorname*{res}_z \frac{1}{z^n} \frac{1}{(1-z)^n} \frac{1}{(1-2z)^2}$$

with $A(z) = \frac{z}{(1-2z)^2}$ and $f(z) = \frac{1}{1-z}$. We get $h(z) = z(1-z)$ and find

$$\operatorname*{res}_w \frac{1}{w^{n+1}} \left[\frac{A(z)}{f(z)h'(z)} \right] \Bigg|_{z=g(w)}.$$

with g the inverse of h.

This becomes

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} \left[\frac{z/(1-2z)^2}{(1-2z)/(1-z)}\right]\Bigg|_{z=g(w)}$$

or alternatively

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}} \left[\frac{z(1-z)}{(1-2z)^3}\right]\Bigg|_{z=g(w)} = \operatorname*{res}_{w} \frac{1}{w^{n}} \left[\frac{1}{(1-2z)^3}\right]\Bigg|_{z=g(w)}.$$

Observe that $(1-2z)^2 = 1-4z+4z^2 = 1-4z(1-z) = 1-4w$ so this is

$$\operatorname*{res}_{w} \frac{1}{w^{n}} \frac{1}{(1-4w)^{3/2}}$$

and the rest of the computation continues as before.

This was math.stackexchange.com problem 4007052.

6 Introductory example for the method, using the Iverson bracket only (I_1)

Suppose we seek to verify that

$$S_n = \sum_{k=0}^{n} 2^{-k} \binom{n+k}{k} = 2^n.$$

We introduce the Iverson bracket

$$[[k \le n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-k+1}} \frac{1}{1-z}\; dz$$

so we may extend k to infinity, getting

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z} \sum_{k\ge 0} 2^{-k} \binom{n+k}{n} z^k \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{n+1}}\; dz.$$

We evaluate this using the negative of the residues at $z=1, z=2$ and $z=\infty$. Here the contour does not include the other two finite poles which also ensures that the geometric series converges. We could choose $\epsilon = 1/2$. We get for the residue at $z=1$

$$-\frac{1}{(1/2)^{n+1}} = -2^{n+1}.$$

For the residue at $z=2$ we write

$$
\begin{aligned}
&(-1)^{n+1}\mathrm{Res}_{z=2}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(z/2-1)^{n+1}}\\
&= (-1)^{n+1}2^{n+1}\mathrm{Res}_{z=2}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(z-2)^{n+1}}\\
&= (-1)^{n}2^{n+1}\mathrm{Res}_{z=2}\frac{1}{(2+(z-2))^{n+1}}\frac{1}{1+(z-2)}\frac{1}{(z-2)^{n+1}}\\
&= (-1)^{n}\mathrm{Res}_{z=2}\frac{1}{(1+(z-2)/2)^{n+1}}\frac{1}{1+(z-2)}\frac{1}{(z-2)^{n+1}}.
\end{aligned}
$$

This is

$$(-1)^n\sum_{q=0}^{n}(-1)^q\binom{n+q}{q}2^{-q}(-1)^{n-q} = \sum_{q=0}^{n}\binom{n+q}{q}2^{-q} = S_n.$$

Finally do the residue at $z = \infty$ getting (this also follows by inspection having degree zero in the numerator and degree $2n+3$ in the denominator)

$$
\begin{aligned}
&\mathrm{Res}_{z=\infty}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(1-z/2)^{n+1}}\\
&= -\mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{1}{1-1/z}\frac{1}{(1-1/2/z)^{n+1}}\\
&= -\mathrm{Res}_{z=0}\frac{1}{z}z^{n+1}\frac{1}{z-1}\frac{z^{n+1}}{(z-1/2)^{n+1}}\\
&= -\mathrm{Res}_{z=0}z^{2n+1}\frac{1}{z-1}\frac{1}{(z-1/2)^{n+1}} = 0.
\end{aligned}
$$

Using the fact that the residues sum to zero we thus obtain

$$S_n - 2^{n+1} + S_n = 0$$

which yields

$$\boxed{S_n = 2^n.}$$

This was math.stackexchange.com problem 389099.

7 Verifying that a certain sum vanishes (B_1)

Suppose we seek to evaluate

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n}\frac{1}{a+bk+1}\binom{a+bk}{m}\binom{k-n-1}{k}.$$

Now when $n = 0$ this becomes $1/(a+1)$ so we will suppose that $n \geq 1$. We also assume that $1/(a+bk+1)$ does not become singular in the range.

Re-write this as

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n} \frac{(-1)^k}{a+bk+1} \binom{a+bk}{m} \binom{n}{k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{a+bk+1} \sum_{m=0}^{n} \binom{n}{m} \binom{a+bk}{m}.$$

We get for the inner sum with $\varepsilon < 1$

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z} (1+z)^{a+bk} \sum_{m=0}^{n} \binom{n}{m} \frac{1}{z^m} \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z} (1+z)^{a+bk} \left(1 + \frac{1}{z}\right)^n \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} (1+z)^{a+bk+n} \; dz = \binom{a+bk+n}{n}.$$

This also follows by Vandermonde. There is the possibility of a pole at $z = -1$ but it is not inside the contour. Substitute into the outer sum to get

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{a+bk+1} \binom{a+bk+n}{n} = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{a+bk+n}{n-1}.$$

With the extractor integral, where $\varepsilon < 1$ since we only want the one binomial coefficient and no cancellation from a potential pole at $z = -1$

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^n} (1+z)^{a+n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^{bk} \; dz$$
$$= \frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^n} (1+z)^{a+n} (1 - (1+z)^b)^n \; dz.$$

This is zero and the sum vanishes, because

$$(1 - (1+z)^b)^n = (-1)^n b^n z^n + \cdots$$

and the third term under the integral cancels the pole at zero. With this last integral when $a + bk + n < 0$ the residue at infinity is zero and when $\varepsilon > 1$ with both poles inside the contour we would get zero for the value of $\binom{a+bk+n}{n-1}$.

This was math.stackexchange.com problem 1789981.

8 A case of radical cancellation (B_1, R)

Suppose we seek to show that

$$\binom{2m}{2n} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{m+k}{2n}.$$

where $m \geq n$. We introduce

$$\binom{2n+1}{2k+1} = \binom{2n+1}{2n-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-2k+1}} (1+z)^{2n+1} \; dz.$$

Observe that this vanishes when $k > n$ so that we may use it to control the range and extend k to infinity. We also use

$$\binom{m+k}{2n} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{2n+1}} (1+w)^{m+k} \; dw.$$

We thus obtain

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \sum_{k \geq 0} z^{2k} (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-(1+w)z^2} \; dw \; dz.$$

Evalute the inner integral using the negative of the residue at the pole at

$$w = \frac{1-z^2}{z^2}$$

(residues sum to zero) as in

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-z^2-wz^2} \; dw \; dz$$

$$= -\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+3}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{w-(1-z^2)/z^2} \; dw \; dz.$$

The negative of the residue is

$$\frac{1}{z^{2m}} \frac{z^{4n+2}}{(1-z^2)^{2n+1}} = \frac{1}{z^{2m-4n-2}} \frac{1}{(1-z^2)^{2n+1}}$$

and we obtain from the outer integral

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+3}} \frac{1}{z^{2m-4n-2}} \frac{1}{(1-z^2)^{2n+1}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-2n+1}} \frac{1}{(1-z)^{2n+1}} \; dz$$
$$= \binom{2m-2n+2n}{2n} = \binom{2m}{2n}.$$

This is the claim.

Remark. We also need to show that the contribution from the residue at infinity of the inner integral is zero. We get

$$\mathrm{Res}_{w=\infty} \frac{(1+w)^m}{w^{2n+1}} \frac{1}{1-(1+w)z^2}$$
$$= -\mathrm{Res}_{w=0} \frac{1}{w^2} (1+1/w)^m w^{2n+1} \frac{1}{1-z^2-z^2/w}$$
$$= -\mathrm{Res}_{w=0} (1+w)^m w^{2n-m} \frac{1}{w(1-z^2)-z^2}.$$

No contribution when $2n \ge m$. Otherwise,

$$\frac{1}{z^2} \mathrm{Res}_{w=0} (1+w)^m \frac{1}{w^{m-2n}} \frac{1}{1-w(1-z^2)/z^2}$$
$$= \frac{1}{z^2} \sum_{q=0}^{m-2n-1} \binom{m}{m-2n-1-q} \frac{(1-z^2)^q}{z^{2q}}$$
$$= \frac{1}{z^2} \sum_{q=0}^{m-2n-1} \binom{m}{2n+1+q} \left(\frac{1}{z^2} - 1\right)^q$$

Combining this with the integral in z yields

$$\sum_{q=0}^{m-2n-1} \binom{m}{2n+1+q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{z^2} \sum_{p=0}^{q} \binom{q}{p} (-1)^{q-p} \frac{1}{z^{2p}} \; dz.$$

The contribution from the residue is

$$[z^{2n+2+2p}](1+z)^{2n+1} = 0.$$

We can express this verbally by saying that the term from the integral is $[z^{2n}](1+z)^{2n+1}$ and the sum only contributes negative powers of z with exponent starting at two.

Remark, II. From the convergence we require that $|z^2(1+w)| < 1$ in the double integral and must choose our contours appropriately. Choosing $\gamma = \epsilon$ we require $\epsilon^2(1+\epsilon) < 1$ so we may take $\epsilon < 1/2$. We must also verify that $(1-z^2)/z^2$ is outside the contour $|w| = \gamma$. This is $1/z^2 - 1$ i.e. a circle of radius $1/\epsilon^2$ shifted by one to the left. The circle has radius at least four so it completely encloses $|w| = \gamma$ from the outside without touching the inner circle, showing that the pole is outside the contour.

This was math.stackexchange.com problem 1900578.

9 Basic usage of exponentiation integral (B_1E)

Suppose we seek to verify that

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q} \binom{q}{r} (-1)^q q^{p-r} = \frac{p!}{r!}.$$

We use the integral representation

$$\binom{q}{r} = \binom{q}{q-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^q}{z^{q-r+1}} \; dz$$

which is zero when $q < r$ (pole vanishes) so we may extend q back to zero. We also use the integral

$$q^{p-r} = \frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(qw)}{w^{p-r+1}} \; dw.$$

We thus obtain for the sum

$$\begin{aligned}
&\frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \\
\times &\frac{1}{2\pi i} \int_{|z|=\epsilon} z^{r-1} \sum_{q=0}^{p} \binom{p}{q} (-1)^q \frac{(1+z)^q}{z^q} \exp(qw) \; dz \; dw \\
&= \frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \\
\times &\frac{1}{2\pi i} \int_{|z|=\epsilon} z^{r-1} \left(1 - \frac{1+z}{z} \exp(w)\right)^p \; dz \; dw \\
&= \frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \\
\times &\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p-r+1}} (-\exp(w) + z(1-\exp(w)))^p \; dz \; dw \\
&= \frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}} \\
\times &\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p-r+1}} (\exp(w) + z(\exp(w)-1))^p \; dz \; dw.
\end{aligned}$$

We extract the residue on the inner integral to obtain

$$\frac{(p-r)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}\binom{p}{p-r}\exp(rw)(\exp(w)-1)^{p-r}\;dw$$
$$=\frac{p!}{r!}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}\exp(rw)(\exp(w)-1)^{p-r}\;dw.$$

It remains to compute

$$[w^{p-r}]\exp(rw)(\exp(w)-1)^{p-r}.$$

Observe that $\exp(w)-1$ starts at w so $(\exp(w)-1)^{p-r}$ starts at w^{p-r} and hence only the constant coefficient from $\exp(rw)$ contributes, the value being one, which finally yields

$$\frac{p!}{r!}.$$

This was math.stackexchange.com problem 1731648.

10 Introductory example for the method, eliminating odd-even dependence (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\binom{k}{\lfloor k/2\rfloor}=\binom{2n+1}{n}.$$

This is

$$\sum_{q=0}^{n}\binom{n}{2q}2^{n-2q}\binom{2q}{q}+\sum_{q=0}^{n}\binom{n}{2q+1}2^{n-2q-1}\binom{2q+1}{q}.$$

We treat these in turn.

First sum. Observe that

$$\binom{n}{2q}\binom{2q}{q}=\binom{n}{q}\binom{n-q}{q}.$$

This yields for the sum

$$2^n\sum_{q=0}^{n}\binom{n}{q}\binom{n-q}{q}2^{-2q}.$$

Introduce

$$\binom{n-q}{q}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-q}}{z^{q+1}}\;dz$$

which yields for the sum

$$\frac{2^n}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z}\sum_{q=0}^{n}\binom{n}{q}2^{-2q}\frac{1}{z^q(1+z)^q}\;dz$$

$$=\frac{2^n}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z}\left(1+\frac{1}{4z(1+z)}\right)^n\;dz$$

$$=\frac{2^{-n}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+2z)^{2n}}{z^{n+1}}\;dz=2^{-n}\binom{2n}{n}2^n=\binom{2n}{n}.$$

Second sum. Observe that

$$\binom{n}{2q+1}\binom{2q+1}{q}=\binom{n}{q}\binom{n-q}{q+1}.$$

This yields for the sum

$$2^{n-1}\sum_{q=0}^{n}\binom{n}{q}\binom{n-q}{q+1}2^{-2q}.$$

This time introduce

$$\binom{n-q}{q+1}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-q}}{z^{q+2}}\;dz$$

which yields for the sum

$$\frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^2}\sum_{q=0}^{n}\binom{n}{q}2^{-2q}\frac{1}{z^q(1+z)^q}\;dz$$

$$=\frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^2}\left(1+\frac{1}{4z(1+z)}\right)^n\;dz$$

$$=\frac{2^{-n-1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+2z)^{2n}}{z^{n+2}}\;dz=2^{-n-1}\binom{2n}{n+1}2^{n+1}=\binom{2n}{n+1}.$$

Conclusion.
Collecting the two contributions we obtain

$$\binom{2n}{n}+\binom{2n}{n+1}=\binom{2n+1}{n}$$

as claimed.
This was math.stackexchange.com problem 1442436.

11 Introductory example for the method, proving equality of two double hypergeometrics (B_1)

Suppose we seek to verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n} \frac{(2k+2v)!}{(k+v)! \times v! \times (2k+v)! \times (n-v)!} 2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{(k+m)! \times m! \times (n-2m)!} 2^{n-4m}.$$

Multiplying by $(n+k)!$ we obtain

$$g_1(n,k) = \sum_{v=0}^{n} \binom{n+k}{n-v} \binom{2k+2v}{v} 2^{-v}$$

and

$$g_2(n,k) = 2^n \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k}{m} \binom{n+k-m}{n-2m} 2^{-4m}.$$

We will work with the latter two.
Re-write the first sum as follows:

$$2^{-n} \sum_{v=0}^{n} \binom{n+k}{v} \binom{2k+2n-2v}{n-v} 2^v$$

Introduce

$$\binom{2k+2n-2v}{n-v} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-v+1}} (1+z)^{2k+2n-2v} \; dz.$$

This integral is zero when $v > n$ so we may extend v to infinity.
We get for $g_1(n,k)$

$$2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2k+2n} \sum_{v \geq 0} \binom{n+k}{v} \frac{z^v}{(1+z)^{2v}} 2^v \; dz$$

$$= 2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2k+2n} \left(1 + 2\frac{z}{(1+z)^2}\right)^{n+k} \; dz$$

$$= 2^{-n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \left(1+4z+z^2\right)^{n+k} \; dz.$$

There are no convergence issues here as the sum in v is in fact finite.
For the second sum introduce

$$\binom{n+k-m}{n-2m} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-2m+1}}(1+z)^{n+k-m}\; dz.$$

This is zero when $2m > n$ so we may extend m to infinity.
We get for $g_2(n,k)$

$$2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{n+k} \sum_{m\ge 0} \binom{n+k}{m} \frac{z^{2m}}{(1+z)^m} 2^{-4m}\; dz$$

$$= 2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{n+k} \left(1+\frac{1}{16}\frac{z^2}{1+z}\right)^{n+k}\; dz$$

$$= 2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \left(1+z+\frac{1}{16}z^2\right)^{n+k}\; dz.$$

Finally put $z = 4w$ in this integral to get

$$2^n \frac{1}{2\pi i}\int_{|w|=\epsilon/4} \frac{1}{4^{n+1}w^{n+1}} \left(1+4w+w^2\right)^{n+k}\; 4dw$$

$$= 2^{-n} \frac{1}{2\pi i}\int_{|w|=\epsilon/4} \frac{1}{w^{n+1}} \left(1+4w+w^2\right)^{n+k}\; dw.$$

This concludes the argument.
This was math.stackexchange.com problem 924966.

12 A remarkable case of factorization (B_1)

We let $T(0) = 0$ and $T(1) = 1$ and prove that when

$$T(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} T(n-k)$$

for $n \ge 2$ then

$$T(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \binom{2n-2}{n-1} - \binom{2n-2}{n}.$$

In fact the case of a zero argument to T is not reached as for $n \ge 2$ we also have $n - \lfloor n/2 \rfloor \ge 1$. Applying the induction hypothesis on the RHS we get two pieces, the first is

$$A = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$
$$= \binom{2n-2}{n-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$

and the second

$$B = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}$$
$$= \binom{2n-2}{n} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}.$$

As we subtract B from A we see that we only need to show that the contribution from the two sum terms call them A' and B' is zero.

For these two pieces we introduce the integral representation

$$\binom{n-k}{k} = \binom{n-k}{n-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-2k+1}} (1+z)^{n-k} \; dz.$$

This has the nice property that it vanishes when $k > \lfloor n/2 \rfloor$ so we may extend the upper limit of the sum to infinity. We also introduce for the first sum

$$\binom{2n-2k-2}{n-k-1} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-k}} (1+w)^{2n-2k-2} \; dw.$$

We thus obtain

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n-2}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^n \sum_{k \ge 0} (-1)^{k+1} \frac{z^{2k} w^k}{(1+z)^k (1+w)^{2k}} \; dz \; dw$$
$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n-2}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^n \frac{1}{1 + z^2 w/(1+z)/(1+w)^2} \; dz \; dw$$
$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{(1+z)(1+w)^2 + z^2 w} \; dz \; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{2n}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}\; dz\; dw.$$

We evaluate the inner integral by summing the residues at $z = -(1+w)$ and $z = -(1+w)/w$ and flipping the sign. (We will verify that the residue at infinity is zero.)

The residue at $z = -(1+w)$ yields

$$-\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{2n}$$

$$\times \frac{(-1)^{n+1}}{(1+w)^{n+1}}(-1)^{n+1}w^{n+1}\frac{1}{-(1+w)+(1+w)/w}\; dw$$

$$= -\frac{1}{2\pi i}\int_{|w|=\gamma} (1+w)^{n-1}\frac{w}{1-w^2}\; dw.$$

This is zero as the pole at zero has been canceled. Next for the residue at $z = -(1+w)/w$ we get

$$-\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{2n}$$

$$\times \frac{(-1)^{n+1}w^{n+1}}{(1+w)^{n+1}}(-1)^{n+1}\frac{1}{w^{n+1}}\frac{1}{-(1+w)/w+1+w}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(1+w)^{n-1}\frac{w}{1-w^2}dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}(1+w)^{n-2}\frac{1}{1-w}dw.$$

With $n \ge 2$ we can evaluate this as

$$\sum_{q=0}^{n-1} {n-2 \choose q} = 2^{n-2}.$$

To wrap up the residue at infinity of the inner integral is

$$\mathrm{Res}_{z=\infty}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}$$

$$= -\mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{1}{1/z+1+w}\frac{1}{1/z+(1+w)/w}$$

$$= -\mathrm{Res}_{z=0}(1+z)^{n+1}\frac{1}{1+z(1+w)}\frac{1}{1+z(1+w)/w} = 0.$$

Collecting everything and flipping the sign we have shown that

$$A' = -2^{n-2}.$$

For piece B' we see that it only differs from A' in an extra $1/w$ factor on the extractor in w at the front. We thus obtain

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+2}}(1+w)^{2n}$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}\;dz\;dw.$$

The residue at $z = -(1+w)$ vanishes the same because there was an extra w to spare on the $w/(1-w^2)$ term:

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{n-1}\frac{1}{1-w^2}\;dw.$$

For the residue at $z = -(1+w)/w$ we are now extracting from

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(1+w)^{n-2}\frac{1}{1-w}dw.$$

to get

$$\sum_{q=0}^{n}\binom{n-2}{q} = 2^{n-2}$$

as before. The residue at infinity vanished in z and did not reach the front extractor in w, for another contribution of zero. This means that

$$B' = -2^{n-2}$$

and we may conclude the proof. The fact that the sum term from the geometric series factored as it did is the remarkable feature of this problem.

Addendum, four years later. In the present version with complex variables the proof requires the convergence of the geometric series. This is $|z^2w/(1+z)/(1+w)^2| < 1$ or $|z^2w| < |(1+z)(1+w)^2|$. Now we have $|(1+z)(1+w)^2| \ge (1-\epsilon)(1-\gamma)^2$ so $(1-\epsilon)(1-\gamma)^2 > \epsilon^2\gamma$ will do. Suppose we take $\epsilon = \gamma$. We obtain $(1-\gamma)^3 > \gamma^3$. Therefore e.g. $\epsilon = \gamma = 1/4$ ensures convergence of the series. This also ensures that the two poles at $-(1+w)$ and $-(1+w)/w$ are outside the contour $|z| = \epsilon$.

This was math.stackexchange.com problem 2113830

13 Evaluating a quadruple hypergeometric(B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{k+l}\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{k}\binom{n}{l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{l=0}^{n}(-1)^l\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{l}.$$

Evaluate the inner sum first and introduce

$$\binom{n+k-l}{n}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k-l}}{z^{n+1}}\;dz.$$

and

$$\binom{k+l}{n}=\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{k+l}}{w^{n+1}}\;dw.$$

This yields for the inner sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(1+w)^l}{(1+z)^l}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}\left(1-\frac{1+w}{1+z}\right)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{k}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^k}{w^{n+1}}(z-w)^n\;dw\;dz.$$

Extracting the inner coefficient yields

$$\sum_{q=0}^{n}\binom{k}{q}\binom{n}{n-q}(-1)^{n-q}z^q.$$

The outer coefficient becomes

$$\sum_{q=0}^{n}\binom{k}{q}\binom{n}{n-q}(-1)^{n-q}\binom{k}{n-q}$$

$$=\sum_{q=0}^{n}\binom{k}{q}\binom{n}{q}(-1)^{n-q}\binom{k}{n-q}.$$

Call this S. By symmetry we have on re-indexing that

$$2S=\sum_{q=0}^{n}\binom{k}{q}\binom{n}{q}((-1)^q+(-1)^{n-q})\binom{k}{n-q}$$

$$= (1+(-1)^n)\sum_{q=0}^{n} \binom{k}{q}\binom{n}{q}(-1)^q \binom{k}{n-q}.$$

This is zero when n is odd so *the entire sum being evaluated vanishes when n is odd* and we may assume that $n = 2m$ and get

$$\sum_{q=0}^{2m} \binom{k}{q}\binom{2m}{q}(-1)^q \binom{k}{2m-q}.$$

Substituting this into the outer sum yields

$$\sum_{q=0}^{2m} \binom{2m}{q}(-1)^q \sum_{k=0}^{2m} \binom{2m}{k}(-1)^k \binom{k}{q}\binom{k}{2m-q}.$$

We evaluate the inner sum with the integrals

$$\binom{k}{q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^k}{z^{q+1}}\; dz.$$

and

$$\binom{k}{2m-q} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^k}{w^{2m-q+1}}\; dw$$

to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} \sum_{k=0}^{2m}\binom{2m}{k}(-1)^k(1+z)^k(1+w)^k\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (z+w+wz)^{2m}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (w(1+z)+z)^{2m}\; dw\; dz.$$

Extracting the coefficient we get for the inner term

$$\binom{2m}{2m-q}(1+z)^{2m-q}z^q$$

and for the outer integral

$$\binom{2m}{2m-q}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}(1+z)^{2m-q}\; dz = \binom{2m}{2m-q}.$$

We are now ready to conclude and return to the main sum which has been transformed into

$$\sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \binom{2m}{2m-q}$$

which is

$$[v^{2m}](1-v)^{2m}(1+v)^{2m} = [v^{2m}](1-v^2)^{2m} = [v^m](1-v)^{2m}$$

$$= (-1)^m \binom{2m}{m}.$$

Observe carefully that there were no convergence issues as both sums were finite, so there is no problem using one and the same value of $\epsilon \ll 1$ in the integrals.

This was math.stackexchange.com problem 1577907.

14 An integral representation of a binomial coefficient involving the floor function (B_1)

Suppose we seek to prove that

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

Observe that from first principles we have that

$$\binom{n}{\lfloor q/2 \rfloor} = \binom{n}{n - \lfloor q/2 \rfloor} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(1 + z + wz^2 + wz^3 + w^2 z^4 + w^2 z^5 + \cdots\right) \; dw \; dz.$$

This simplifies to

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(\frac{1}{1-wz^2} + z \frac{1}{1-wz^2} \right) \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw \; dz.$$

This correctly enforces the range as the reader is invited to verify and we may extend k beyond $2m+1$, getting for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \sum_{k\ge 0} {n\choose k} 2^k z^k \frac{w^k}{(1+w)^k} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \left(1+\frac{2wz}{1+w}\right)^n \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w+2wz)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw\; dz.$$

Extracting the inner coefficient now yields

$$\sum_{q=0}^n {n\choose q} (1+2z)^q z^{2n-2q} = z^{2n} \sum_{q=0}^n {n\choose q} (1+2z)^q z^{-2q}$$

$$= z^{2n} \left(1+\frac{1+2z}{z^2}\right)^n = (1+z)^{2n}.$$

We thus get from the outer coefficient

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2m+2}} \; dz$$

which is

$${2n+1\choose 2m+1}$$

as claimed. I do believe this is an instructive exercise.
This was math.stackexchange.com problem 2087559

15 Evaluating another quadruple hypergeometric

Suppose we seek to verify that

$$\sum_{k=m}^n (-1)^{n+k} \frac{2k+1}{n+k+1} {n\choose k} {n+k\choose k}^{-1} {k\choose m} {k+m\choose m} = \delta_{mn}.$$

Here we may assume $n\ge m$, the equality holds trivially otherwise.
Now we have

$${n\choose k} {n+k\choose k}^{-1} = \frac{n!}{k!(n-k)} \frac{k!n!}{(n+k)!}$$

$$= \frac{n!}{(n-k)} \frac{n!}{(n+k)!} = \binom{2n}{n+k} \binom{2n}{n}^{-1}.$$

We get for the sum

$$\sum_{k=m}^{n} (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{2n}{n+k} \binom{k}{m} \binom{k+m}{m} = \delta_{mn} \times \binom{2n}{n}.$$

which is

$$\sum_{k=m}^{n} (-1)^{n+k} (2k+1) \binom{2n+1}{n+k+1} \binom{k}{m} \binom{k+m}{m}$$

$$= \delta_{mn} \times (2n+1) \times \binom{2n}{n}.$$

We also have

$$\binom{k}{m} \binom{k+m}{m} = \frac{(k+m)!}{m! \times (k-m)! \times m!} = \binom{2m}{m} \binom{k+m}{k-m}.$$

The target statement to prove now becomes

$$\binom{2m}{m} \sum_{k=m}^{n} (-1)^{n+k} (2k+1) \binom{2n+1}{n+k+1} \binom{k+m}{k-m}$$

$$= \delta_{mn} \times (2n+1) \times \binom{2n}{n}.$$

With the LHS in mind introduce

$$\binom{2n+1}{n+k+1} = \binom{2n+1}{n-k} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n-k+1}} (1+z)^{2n+1} \, dz.$$

Observe that this vanishes when $k > n$ so we may extend k upward to infinity. Furthermore introduce

$$\binom{k+m}{2m} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{2m+1}} (1+w)^{k+m} \, dw.$$

Observe once again that the integral vanishes, this time when $0 \le k < m$ so we may extend k back to zero.

We then have for the sum without the scalar $\binom{2m}{m}$

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{2n+1}}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2m+1}}$$

$$\times \sum_{k \ge 0} (-1)^k (2k+1) z^k (1+w)^k \, dw \, dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{2n+1}}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2m+1}} \times \frac{1-z(1+w)}{(1+z(1+w))^2} \; dw \; dz.$$

This is

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{2n+1}}{z^{n+3}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w^{2m+1}} \times \frac{1-z(1+w)}{(w+(1+z)/z)^2} \; dw \; dz.$$

Here the convergence of the geometric series requires $|z(1+w)| < 1$. We have $|z(1+w)| \le \varepsilon(1+\gamma)$ so we can take $\gamma = \varepsilon$ and $\varepsilon < 1/2$. We will now work with the pole at $w = -(1+z)/z$. We have for this pole the norm $|1/z+1| \ge \varepsilon^{-1} - 1 > \gamma$ because $1 - \varepsilon > \varepsilon^2$ from the convergence requirement. This means the pole is outside the contour in γ. With residues summing to zero and the residue at infinity being zero by inspection we can evaluate the inner integral as minus the contribution from the residue at that second finite pole.

We get for the first piece,

$$\left[m \frac{(1+w)^{m-1}}{w^{2m+1}} - (2m+1) \frac{(1+w)^m}{w^{2m+2}} \right] \Bigg|_{w=-(1+z)/z}$$
$$= m(-1)^{2m+1} \frac{z^{2m+1}}{(1+z)^{2m+1}} (-1)^{m-1} \frac{1}{z^{m-1}}$$
$$-(2m+1)(-1)^{2m+2} \frac{z^{2m+2}}{(1+z)^{2m+2}} (-1)^m \frac{1}{z^m}.$$

Extracting the residue on z yields

$$(-1)^m m \binom{2n-2m}{n-m} - (-1)^m (2m+1) \binom{2n-2m-1}{n-m}.$$

Continuing with the second piece, which has the factor $z(1+w)$

$$\left[(m+1) \frac{(1+w)^m}{w^{2m+1}} - (2m+1) \frac{(1+w)^{m+1}}{w^{2m+2}} \right] \Bigg|_{w=-(1+z)/z}$$
$$= (m+1)(-1)^{2m+1} \frac{z^{2m+1}}{(1+z)^{2m+1}} (-1)^m \frac{1}{z^m}$$
$$-(2m+1)(-1)^{2m+2} \frac{z^{2m+2}}{(1+z)^{2m+2}} (-1)^{m+1} \frac{1}{z^{m+1}}.$$

Once more extracting the residue on z yields

$$-(-1)^m(m+1)\binom{2n-2m}{n-m} + (-1)^m(2m+1)\binom{2n-2m-1}{n-m}.$$

Subtract the second piece from the first,

$$(-1)^m(2m+1)\binom{2n-2m}{n-m} - (-1)^m(4m+2)\binom{2n-2m-1}{n-m}.$$

Now when $n = m$ this evaluates to $-(-1)^m(2m+1)$ and when $n > m$ we get

$$(-1)^m(2m+1)2\binom{2n-2m-1}{n-m-1} - (-1)^m(4m+2)\binom{2n-2m-1}{n-m} = 0.$$

Taking into account the sign flip from the residue we are left with

$$\delta_{nm}\binom{2m}{m}(-1)^n(-1)^m(2m+1) = \delta_{nm}(2n+1)\binom{2n}{n}$$

which holds by inspection.
This was math.stackexchange.com problem 1817122.

16 An identity by Strehl (B_1)

Suppose we seek to show that

$$\sum_{k=0}^{n}\binom{n}{k}^3 = \sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}^2\binom{2k}{n}.$$

With

$$\binom{n}{k}\binom{2k}{n} = \frac{(2k)!}{k! \times (n-k)! \times (2k-n)!} = \binom{2k}{k}\binom{k}{n-k}$$

we find that the RHS is

$$\sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}\binom{2k}{k}\binom{k}{n-k}.$$

Introduce

$$\binom{2k}{k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2k}}{z^{k+1}}\;dz$$

and (this integral is zero when $0 \le k < \lceil n/2\rceil$)

$$\binom{k}{n-k} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^k}{w^{n-k+1}}\;dw$$

to get for the RHS

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{w^k(1+w)^k(1+z)^{2k}}{z^k}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\left(1+\frac{w(1+w)(1+z)^2}{z}\right)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(z+w(1+w)(1+z)^2)^n\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(z+w(z+1))^n(1+w(z+1))^n\;dw\;dz.$$

Extracting first the residue in w in next the residue in z we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\sum_{q=0}^{n}\binom{n}{q}z^{n-q}(1+z)^q\binom{n}{n-q}(1+z)^{n-q}\;dz$$

$$=\sum_{q=0}^{n}\binom{n}{q}^2\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{q+1}}\;dz$$

$$=\sum_{q=0}^{n}\binom{n}{q}^3$$

QED.

Addendum May 27 2018. We compute this using formal power series as per request in comment. Start from

$$\binom{2k}{k}=[z^k](1+z)^{2k}$$

and

$$\binom{k}{n-k}=[w^{n-k}](1+w)^k.$$

Observe that this coefficient extractor is zero when $n-k>k$ or $k<\lceil n/2\rceil$ where $k\ge 0$. Hence we are justified in lowering k to zero when we substitute these into the sum and we find

$$\sum_{k=0}^{n}\binom{n}{k}[z^k](1+z)^{2k}[w^{n-k}](1+w)^k$$

$$=[z^0][w^n]\sum_{k=0}^{n}\binom{n}{k}\frac{1}{z^k}(1+z)^{2k}w^k(1+w)^k$$

$$= [z^0][w^n]\left(1+\frac{(1+z)^2w(1+w)}{z}\right)^n$$

$$= [z^n][w^n](z+(1+z)^2w(1+w))^n$$

$$= [z^n][w^n](1+w(1+z))^n(z+w(1+z))^n.$$

We extract the coefficient on $[w^n]$ then the one on $[z^n]$ and get

$$[z^n]\sum_{q=0}^{n}\binom{n}{q}(1+z)^q\binom{n}{n-q}(1+z)^{n-q}z^q$$

$$=\sum_{q=0}^{n}\binom{n}{q}^2[z^{n-q}](1+z)^n=\sum_{q=0}^{n}\binom{n}{q}^2\binom{n}{n-q}=\sum_{q=0}^{n}\binom{n}{q}^3.$$

The claim is proved.
This was math.stackexchange.com problem 586138.

17 A simple triple hypergeometric

We seek to verify with $n \ge k, \ell \ge 0$ that

$$\sum_{s\ge 0}\binom{n+s}{k+\ell}\binom{k}{s}\binom{\ell}{s}=\binom{n}{k}\binom{n}{\ell}.$$

We get with the usual extractors,

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^n}{z^{k+\ell+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^\ell}{w^{\ell+1}}\sum_{s\ge 0}\binom{k}{s}(1+z)^s w^s\; dw\; dz.$$

Here we have $\varepsilon, \gamma \ll 1$ and we may evalate the sum without regard to convergence issues because it contains a finite number of terms. Doing so will produce

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^n}{z^{k+\ell+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^\ell}{w^{\ell+1}}[1+(1+z)w]^k\; dw\; dz.$$

Expanding the binomial as

$$\sum_{q=0}^{k}\binom{k}{q}(1+w)^q z^{k-q} w^{k-q}$$

yields

$$\sum_{q=0}^{k}\binom{k}{q}\binom{\ell+q}{\ell+q-k}\binom{n}{\ell+q}.$$

Next observe that

$$\binom{\ell+q}{k}\binom{n}{\ell+q} = \frac{n!}{(n-\ell-q)! \times k! \times (\ell+q-k)!} = \binom{n}{k}\binom{n-k}{\ell+q-k}.$$

We have obtained the first factor $\binom{n}{k}$. This leaves

$$\sum_{q=0}^{k}\binom{k}{q}\binom{n-k}{\ell+q-k} = \sum_{q=0}^{k}\binom{k}{q}\binom{n-k}{n-\ell-q}$$

$$= \frac{1}{2\pi i}\int_{|v|=\rho} \frac{(1+v)^{n-k}}{v^{n-\ell+1}} \sum_{q=0}^{k}\binom{k}{q} v^q \; dv$$

$$= \frac{1}{2\pi i}\int_{|v|=\rho} \frac{(1+v)^{n-k}}{v^{n-\ell+1}} (1+v)^k \; dv$$

$$= \frac{1}{2\pi i}\int_{|v|=\rho} \frac{(1+v)^{n}}{v^{n-\ell+1}} \; dv = \binom{n}{n-\ell} = \binom{n}{\ell}.$$

This is the second factor and we may conclude. The second sum also yields to Vandermonde seeing as if $k < n-\ell$ we may raise q to $n-\ell$ due to the first binomial coefficient and if $k > n-\ell$ we may lower to $n-\ell$ due to the second one.

This was math.stackexchange.com problem 2381429.

18 Working with negative indices (B_1)

Suppose we seek to prove that

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k \binom{2n}{n+3k} = 2 \times 3^{n-1}.$$

We start by introducing the integral

$$\binom{2n}{n+3k} = \binom{2n}{n-3k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-3k+1}}(1+z)^{2n} \; dz.$$

Observe that this vanishes for $3k > n$ (pole canceled) and for $3k < -n$ (upper range of polynomial term exceeded) so we may extend the summation to $[-n, n]$ getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}(1+z)^{2n} \sum_{k=-n}^{n} (-1)^k z^{3k} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n} (-1)^n z^{-3n} \sum_{k=0}^{2n} (-1)^k z^{3k} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n} (-1)^n \frac{1-(-1)^{2n+1} z^{3(2n+1)}}{1+z^3} \; dz.$$

Only the first piece from the difference due to the geometric series contributes and we get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n} (-1)^n \frac{1}{1+z^3} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{4n+1}} (1+z)^{2n-1} (-1)^n \frac{1}{1-z+z^2} \; dz.$$

We have two poles other than zero and infinity at ρ and $1/\rho$ where

$$\rho = \frac{1+\sqrt{3}i}{2}$$

and using the fact that residues sum to zero we obtain

$$S + \frac{(-1)^n}{\rho(1+\rho)} \frac{1}{\rho - 1/\rho} \left(\frac{(1+\rho)^2}{\rho^4} \right)^n + \frac{(-1)^n}{1/\rho(1+1/\rho)} \frac{1}{1/\rho - \rho} \left(\frac{(1+1/\rho)^2}{1/\rho^4} \right)^n$$

$$+ \mathrm{Res}_{z=\infty} \frac{1}{z^{4n+1}} (1+z)^{2n-1} (-1)^n \frac{1}{1-z+z^2} = 0.$$

We get for the residue at infinity

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{4n+1} (1+1/z)^{2n-1} (-1)^n \frac{1}{1-1/z+1/z^2}$$

$$= -\mathrm{Res}_{z=0} z^{2n+2} (1+z)^{2n-1} (-1)^n \frac{1}{z^2-z+1} = 0.$$

Now if $z^2 = z-1$ then $z^4 = z^2 - 2z + 1 = -z$ and thus

$$\frac{(1+1/\rho)^2}{1/\rho^4} = \frac{(1+\rho)^2}{\rho^4} = \frac{\rho - 1 + 2\rho + 1}{-\rho} = -3$$

and furthermore with $z(1+z)(z-1/z) = (1+z)(z^2-1)$ and $(1+z)(z-2) = z^2 - z - 2 = -3$ we finally get

$$S + (-1)^n \times \left(-\frac{1}{3}\right) (-3)^n + (-1)^n \times \left(-\frac{1}{3}\right) (-3)^n = 0$$

or

$$S = 2 \times 3^{n-1}.$$

This was math.stackexchange.com problem 2054777.

19 Two companion identities by Gould (B_1)

Suppose we seek to evaluate

$$Q(x,\rho) = \sum_{k=0}^{\rho} \binom{2x+1}{2k} \binom{x-k}{\rho-k}$$

where $x \ge \rho$.
Introduce

$$\binom{x-k}{\rho-k} = \binom{x-k}{x-\rho} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} (1+z)^{x-k} \; dz.$$

Note that this controls the range being zero when $\rho < k \le x$ so we can extend the sum to x supposing that $x > \rho$. And when $x = \rho$ we may also set the upper limit to x.

We get for the sum

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} (1+z)^x \sum_{k=0}^{x} \binom{2x+1}{2k} \frac{1}{(1+z)^k} \; dz.$$

This is

$$\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} (1+z)^x \left(\left(1 + \frac{1}{\sqrt{1+z}}\right)^{2x+1} + \left(1 - \frac{1}{\sqrt{1+z}}\right)^{2x+1} \right) \; dz$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} \frac{1}{\sqrt{1+z}} \left((1+\sqrt{1+z})^{2x+1} + (1-\sqrt{1+z})^{2x+1} \right) \; dz.$$

Observe that the second term in the parenthesis (i.e. $1 - \sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution. This leaves

$$\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} \frac{1}{\sqrt{1+z}} (1+\sqrt{1+z})^{2x+1} \; dz.$$

Here we have $\varepsilon \ll 1$ so the pole at $z = -1$ is not inside the contour. Now put $1 + z = w^2$ so that $dz = 2w \; dw$ to get

$$\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w^2-1)^{x-\rho+1}} \frac{1}{w} (1+w)^{2x+1} \; w \; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho+1}} \frac{1}{(w+1)^{x-\rho+1}} (1+w)^{2x+1} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho+1}} (1+w)^{x+\rho} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho+1}} \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q} (w-1)^q \; dw.$$

Here we use the principal branch of the logarithm as in $\sqrt{1+z} = \exp(\frac{1}{2}\log(1+z))$ with branch cut $(-\infty, -1]$. We have for $|z| < 1$ that $\sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \cdots$.

This means that the image of $|z| = \varepsilon$ under this substitution approximates the circle $|w-1| = \frac{1}{2}\varepsilon$, making one turn. It contains the one pole at $w = 1$. With the radius of convergence ρ being one we obtain $|w-1| = \frac{1}{2}\varepsilon + O(\varepsilon^2)$ from the geometric series $(\varepsilon/\rho)^2/(1-\varepsilon/\rho)$. The image approximates the circle as closely as we wish as ε is small. Hence we may take $\gamma = \frac{1}{2}\varepsilon$. Continuing we find

$$[(w-1)^{x-\rho}] \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q} (w-1)^q$$

$$= \binom{x+\rho}{x-\rho} 2^{x+\rho-(x-\rho)} = \binom{x+\rho}{x-\rho} 2^{2\rho} = \binom{x+\rho}{2\rho} 2^{2\rho}.$$

We can also prove the companion identity from above.

Suppose we seek to evaluate

$$Q(x, \rho) = \sum_{k=0}^{\rho} \binom{2x+1}{2k+1} \binom{x-k}{\rho-k}$$

where $x \geq \rho$. With the same preliminaries as before we obtain

$$\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^x}{z^{x-\rho+1}} \sqrt{1+z} \left(\left(1 + \frac{1}{\sqrt{1+z}}\right)^{2x+1} - \left(1 - \frac{1}{\sqrt{1+z}}\right)^{2x+1} \right) \; dz$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} \left((1+\sqrt{1+z})^{2x+1} - (1-\sqrt{1+z})^{2x+1} \right) \; dz.$$

We observe once more that the second term in the parenthesis (i.e. $1 - \sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution, which leaves

$$\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{x-\rho+1}} (1+\sqrt{1+z})^{2x+1} \; dz.$$

We again put $1 + z = w^2$ so that $dz = 2w\, dw$ to get

$$\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w^2-1)^{x-\rho+1}} (1+w)^{2x+1}\; w\; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho+1}} \frac{1}{(w+1)^{x-\rho+1}} (1+w)^{2x+1}\; w\; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho+1}} (1+w)^{x+\rho}\; w\; dw.$$

Writing $w = (w-1) + 1$ this produces two pieces, the first is

$$\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{x-\rho}} \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q} (w-1)^q\; dw.$$

This is

$$[(w-1)^{x-\rho-1}] \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q} (w-1)^q$$

$$= \binom{x+\rho}{x-\rho-1} 2^{x+\rho-(x-\rho-1)} = \binom{x+\rho}{x-\rho-1} 2^{2\rho+1} = \binom{x+\rho}{2\rho+1} 2^{2\rho+1}.$$

The second piece is

$$[(w-1)^{x-\rho}] \sum_{q=0}^{x+\rho} \binom{x+\rho}{q} 2^{x+\rho-q} (w-1)^q$$

$$= \binom{x+\rho}{x-\rho} 2^{x+\rho-(x-\rho)} = \binom{x+\rho}{x-\rho} 2^{2\rho} = \binom{x+\rho}{2\rho} 2^{2\rho}.$$

Joining the two pieces we finally obtain

$$\left(2 \times \frac{x-\rho}{2\rho+1} + 1\right) \times \binom{x+\rho}{2\rho} 2^{2\rho}$$

$$= \frac{2x+1}{2\rho+1} \binom{x+\rho}{2\rho} 2^{2\rho}.$$

This was math.stackexchange.com problem 1383343.

20 Exercise 1.3 from Stanley's Enumerative Combinatorics

Suppose we seek to verify that

$$\sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k}\binom{x}{b-k}\binom{y}{a-k} = \binom{x+a}{b}\binom{y+b}{a}$$

where we take $y \geq a$ and $x \geq b$ both integers We then have it for all x and y as both sides are polynomials in those variables. We also take $a, b \geq 0$.

We have for the LHS with the usual extractors,

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^x}{z^{b+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^y}{w^{a+1}} \sum_{k\geq 0} \binom{x+y+k}{k} z^k w^k \; dw \; dz.$$

Here we have extended the range of the sum to infinity because the two residues combined enforce the upper limit of $\min(a, b)$. Continuing,

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^x}{z^{b+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^y}{w^{a+1}} \frac{1}{(1-zw)^{x+y+1}} \; dw \; dz.$$

Now for the geometric series to converge we take $\varepsilon, \gamma \ll 1$. We re-write

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^x}{z^{b+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^y}{w^{a+1}} \frac{1}{(1+z-z(1+w))^{x+y+1}} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{b+1}} \frac{1}{(1+z)^{y+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^y}{w^{a+1}} \frac{1}{(1-z(1+w)/(1+z))^{x+y+1}} \; dw \; dz.$$

Expanding the fractional term taking the residue in z into account we find

$$\sum_{q=0}^{b} \binom{x+y+q}{q} z^q \frac{(1+w)^q}{(1+z)^q}.$$

Substitute into the integrals to get

$$\sum_{q=0}^{b} \binom{x+y+q}{q} (-1)^{b-q} \binom{y+b}{b-q}\binom{y+q}{a}.$$

Note that

$$\binom{y+b}{b-q}\binom{y+q}{a} = \frac{(y+b)!}{(b-q)! \times a! \times (y+q-a)!} = \binom{y+b}{a}\binom{y+b-a}{b-q}.$$

We have obtained the first factor on the RHS. This leaves

$$\sum_{q=0}^{b} \binom{x+y+q}{q} (-1)^{b-q} \binom{y+b-a}{b-q}$$
$$= \frac{1}{2\pi i} \int_{|v|=\rho} \frac{1}{v^{b+1}} (1+v)^{y+b-a} \sum_{q\ge 0} \binom{x+y+q}{q} (-1)^{b-q} v^q \; dv.$$

Here we have extended to infinity due to the residue in v and we take $\rho \ll 1$. We finally have

$$(-1)^b \frac{1}{2\pi i} \int_{|v|=\rho} \frac{1}{v^{b+1}} (1+v)^{y+b-a} \frac{1}{(1+v)^{x+y+1}} \; dv$$
$$= (-1)^b \binom{b-a-x-1}{b} = \binom{x+a}{b}.$$

This is the second factor which concludes the argument. Observe that when we expanded into the alternate series convergence requires $|z(1+w)/(1+z)| < 1$. The norm is bounded by $\varepsilon(1+\gamma)/(1-\varepsilon)$ so we need $\varepsilon(2+\gamma) < 1$ or $\varepsilon < 1/(2+\gamma)$. For example $\varepsilon = 1/5$ and $\gamma = 1/2$ will work.

This was math.stackexchange.com problem 1426447.

21 Counting m-subsets (B_1)

Suppose we seek to verify that

$$\sum_{q=0}^{n} \binom{n}{2q} \binom{n-2q}{p-q} 2^{2q} = \binom{2n}{2p}.$$

Observe that the sum is

$$\sum_{q=0}^{n} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q$$

which is

$$\sum_{q=0}^{p} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q = 4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p} 4^{-q}$$

$$= 4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{2p-2q} 4^{-q}.$$

Now introduce

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p-2q+1}} (1+z)^{n-q} \; dz$$

This is zero when $q > p$ so it provides the range control. Continuing we get for the sum

$$4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \sum_{q\ge 0} \binom{n}{q} 4^{-q} \frac{z^{2q}}{(1+z)^q} \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \left(1 + \frac{1}{4}\frac{z^2}{1+z}\right)^n \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} \left(1 + z + \frac{1}{4} z^2\right)^n \; dz$$

Now put $z = 2w$ to get

$$4^p \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{2^{2p+1} w^{2p+1}} \left(1 + 2w + w^2\right)^n \; 2dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{2p+1}} (1+w)^{2n} \; dw.$$

This is

$$\binom{2n}{2p}$$

as claimed. Here we take $\gamma = 2\epsilon < 1$. This was math.stackexchange.com problem 1430202.

22 Method applied to an iterated sum ($B_1 R$)

Suppose we seek to show that

$$\sum_{k=0}^{n-1} \left(\sum_{q=0}^{k} \binom{n}{q} \right) \left(\sum_{q=k+1}^{n} \binom{n}{q} \right) = \frac{1}{2} n \binom{2n}{n}.$$

Using the integral representation

$$\binom{n}{q} = \binom{n}{n-q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n-q+1}} \; dz$$

where we shall see that only finite sums appear and there are no convergence issues, we get for the first factor

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \sum_{q=0}^{k} z^q \; dz = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1 - z^{k+1}}{1-z} \; dz$$

$$= 2^n - \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{z^{k+1}}{1-z} \; dz$$

and for the second factor

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{z^{k+1}-z^{n+1}}{1-z}\;dz = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{z^{k+1}}{1-z}\;dz.$$

These add to 2^n as they obviously should.

Summing from $k=0$ to $n-1$ we get a positive and a negative piece. The positive piece is

$$2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\sum_{k=0}^{n-1}\frac{z^k}{1-z}\;dz$$
$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\frac{1-z^n}{(1-z)^2}\;dz$$
$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n}\frac{1}{(1-z)^2}\;dz.$$

The negative piece is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n(1-z)}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^n}{w^n(1-w)}\sum_{k=0}^{n-1}z^k w^k\;dw\;dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n(1-z)}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^n}{w^n(1-w)}\frac{1-z^n w^n}{1-zw}\;dw\;dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n(1-z)}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^n}{w^n(1-w)}\frac{1}{1-zw}\;dw\;dz.$$

We evaluate the inner integral by taking the sum of the negatives of the residues of the poles at $w=1$ and $w=1/z$ instead of computing the residue of the pole at zero by using the fact that the residues sum to zero.

Re-write the integral as follows.

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^n}{w^n(w-1)}\frac{1}{zw-1}\;dw$$
$$= \frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^n}{w^n(w-1)}\frac{1}{w-1/z}\;dw.$$

Now the negative of the residue at $w=1$ is

$$-\frac{1}{z}2^n\frac{1}{1-1/z} = 2^n\frac{1}{1-z}.$$

Substituting this into the outer integral we get

$$2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^n(1-z)^2}\;dz.$$

We see that this piece precisely cancels the positive piece that we obtained first.

Continuing the negative of the residue at $w = 1/z$ is

$$-\frac{1}{z}\frac{(1+1/z)^n}{1/z^n \times (1/z-1)} = -\frac{1}{z}\frac{(1+z)^n}{(1/z-1)} = -\frac{(1+z)^n}{(1-z)}.$$

We now substitute this into the outer integral flipping the sign because this was the negative piece to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^n(1-z)^2}\; dz.$$

Extracting the residue at $z = 0$ we get

$$\sum_{q=0}^{n-1} \binom{2n}{n-1-q}(q+1) = \sum_{q=0}^{n-1} \binom{2n}{n+q+1}(q+1)$$

$$= -n\sum_{q=0}^{n-1} \binom{2n}{n+q+1} + \sum_{q=0}^{n-1} \binom{2n}{n+q+1}(n+q+1)$$

$$= -n\left(\frac{1}{2}2^{2n} - \frac{1}{2}\binom{2n}{n}\right) + 2n\sum_{q=0}^{n-1}\binom{2n-1}{n+q}$$

$$= -n\left(\frac{1}{2}2^{2n} - \frac{1}{2}\binom{2n}{n}\right) + 2n\frac{1}{2}2^{2n-1}$$

$$= \frac{1}{2}n\binom{2n}{n}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the inner integral is zero. We use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0}\left[-\frac{1}{z^2}h\left(\frac{1}{z}\right)\right]$$

which in the present case gives for the inner term in w

$$-\mathrm{Res}_{w=0}\frac{1}{w^2}\frac{(1+1/w)^n}{1/w^n \times (1-1/w)}\frac{1}{1-z/w}$$

$$= -\mathrm{Res}_{w=0}\frac{1}{w^2}\frac{(1+w)^n}{(1-1/w)}\frac{1}{1-z/w}$$

$$= -\mathrm{Res}_{w=0}\frac{(1+w)^n}{(w-1)}\frac{1}{w-z}$$

which is zero by inspection. Here z runs on the circle $|z| = \epsilon > 0$.

This was math.stackexchange.com problem 889892.

23 A pair of two double hypergeometrics (B_1)

We seek to show that

$$(1-x)^{2n+1} \sum_{j\ge 0} \binom{j+n-1}{n}\binom{j+n}{n} x^j = \sum_{j\ge 0} \binom{n-1}{j-1}\binom{n+1}{j} x^j.$$

where we set $n \ge 1$ for the parameter. We obtain for the coefficient on $[x^j]$ of the LHS that it is

$$\sum_{q=0}^{j} \binom{2n+1}{q} (-1)^q \binom{j-q+n}{n}\binom{j-q+n-1}{n}.$$

With the usual coefficient extractor integrals,

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{j+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^j}\frac{1}{(1-w)^{n+1}} \\ \times \sum_{q\ge 0} \binom{2n+1}{q} (-1)^q z^q w^q \; dw \; dz.$$

Here we have removed the upper limit on q because the residue in z vanishes when $q > j$. Note that the sum is finite, so we may just deploy the usual $\varepsilon, \gamma < 1$. We get

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{j+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^j}\frac{1}{(1-w)^{n+1}} (1-zw)^{2n+1} \; dw \; dz.$$

Now write

$$(1-zw)^{2n+1} = (1-z+(1-w)z)^{2n+1} = \sum_{q=0}^{2n+1} \binom{2n+1}{q} (1-z)^{2n+1-q}(1-w)^q z^q.$$

Extracting the residues we obtain using the fact that the residue in z vanishes when $q > j$

$$\sum_{q=0}^{j} \binom{2n+1}{q}\binom{n-q}{j-q} (-1)^{j-q} \binom{n-q+j-1}{j-1}.$$

Now observe that

$$\binom{n-q}{j-q}\binom{n-q+j-1}{n-q} = \frac{(n-q+j-1)!}{(j-q)! \times (n-j)! \times (j-1)!}$$
$$= \binom{n-1}{j-1}\binom{n-q+j-1}{n-1}.$$

We have obtained the first factor of the closed form coefficient, leaving us with

$$\sum_{q=0}^{j} \binom{2n+1}{q} (-1)^{j-q} \binom{n-q+j-1}{j-q}$$

which is

$$\frac{1}{2\pi i} \int_{|v|=\rho} \frac{1}{v^{j+1}} (1+v)^{n+j-1} \sum_{q \ge 0} \binom{2n+1}{q} (-1)^{j-q} \frac{v^q}{(1+v)^q} \; dv.$$

Here we have again removed the upper limit on the sum because the residue at zero will produce zero when $q > j$. Note also that when $q > n+j-1$ a pole appears at $v = -1$. We don't want it to contribute so we choose $\rho < 1$. Continuing,

$$\frac{(-1)^j}{2\pi i} \int_{|v|=\rho} \frac{1}{v^{j+1}} (1+v)^{n+j-1} (1 - v/(1+v))^{2n+1} \; dv$$
$$= \frac{(-1)^j}{2\pi i} \int_{|v|=\rho} \frac{1}{v^{j+1}} \frac{1}{(1+v)^{n-j+2}} \; dv$$
$$= (-1)^j (-1)^j \binom{n-j+1+j}{j} = \binom{n+1}{j}.$$

This is the second factor, which concludes the argument. The sums were finite.

This was math.stackexchange.com problem 869982.

24 A two phase application of the method (B_1)

We seek to show that

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-3k}{n} = \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n+1}{k} \binom{k}{n-k}.$$

Note that the second binomial coefficient in both sums controls the range of the sum, so we can write our claim like this:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \binom{2n-3k}{n-3k} = \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{k}{n-k}.$$

To evaluate the LHS introduce the integral representation

$$\binom{2n-3k}{n-3k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-3k}}{z^{n-3k+1}} \; dz.$$

We can check that this really is zero when $k > \lfloor n/3 \rfloor$.

This gives for the sum the representation

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \left(\frac{z^3}{(1+z)^3} \right)^k \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \left(1 - \frac{z^3}{(1+z)^3} \right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \left(3z^2 + 3z + 1\right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^q (1+z)^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^{q-n-1} (1+z)^{q-n-3} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q \frac{1}{z^{n+1-q}} \frac{1}{(1+z)^{n+3-q}} \; dz.$$

Computing the residue we find

$$\sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{n-q+n+2-q}{n+2-q}$$

$$= \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{2n-2q+2}{n-q+2}.$$

Now introduce the integral representation

$$\binom{2n-2q+2}{n-q+2} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-2q+2}}{z^{n-q+3}} \; dz$$

which gives for the sum the integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+3}}\sum_{q=0}^{n+1}\binom{n+1}{q}3^q(-1)^{n-q}\left(\frac{z}{(1+z)^2}\right)^q\;dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+3}}\left(\frac{3z}{(1+z)^2}-1\right)^{n+1}\;dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+3}}(-1+z-z^2)^{n+1}\;dz.$$

Put $w=-z$ which just rotates the small circle to get

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(-w)^{n+3}}(-1-w-w^2)^{n+1}\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+3}}(1+w+w^2)^{n+1}\;dw.$$

We get for the final answer

$$[w^{n+2}](1+w+w^2)^{n+1}$$

but we have $2n+2-n-2=n$ and thus exploiting the symmetry of $1+w+w^2$ we get

$$[w^n](1+w+w^2)^{n+1}.$$

To evaluate the RHS introduce the integral representation

$$\binom{k}{n-k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^k}{z^{n-k+1}}\;dz.$$

This gives for the sum the representation

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}((1+z)z)^k\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z(1+z))^{n+1}\;dz.$$

The answer is

$$[z^n](1+z+z^2)^{n+1},$$

the same as the LHS, and we are done.

This was math.stackexchange.com problem 664823.

25 An identity from Mathematical Reflections (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor (m+n)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Observe that in the second binomial coefficient we must have $m+n-2k \geq n-1$ in order to avoid hitting the zero value in the product in the numerator of the binomial coefficient, so the upper limit for the sum is in fact $m+1 \geq 2k$ with the sum being

$$\sum_{k=0}^{\lfloor (m+1)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Introduce

$$\binom{m+n-2k}{n-1} = \binom{m+n-2k}{m+1-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n-2k}}{z^{m+2-2k}} \; dz.$$

This integral correctly encodes the range for k being zero when k is larger than $\lfloor (m+1)/2 \rfloor$. Therefore we may let k go to infinity in the sum and obtain for $n > m$

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \sum_{k \geq 0} \binom{n}{k} (-1)^k \frac{z^{2k}}{(1+z)^{2k}} \; dz \\ &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \left(1 - \frac{z^2}{(1+z)^2}\right)^n \; dz \\ &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{(1+z)^{n-m} z^{m+2}} (1+2z)^n \; dz. \end{aligned}$$

This produces the closed form

$$\sum_{q=0}^{m+1} \binom{n}{q} 2^q (-1)^{m+1-q} \binom{m+1-q+n-m-1}{n-m-1}$$

$$= (-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{n-m-1}.$$

This is

$$(-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{m+1-q}.$$

Introduce
$$\binom{n-q}{m+1-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{m+2-q}} \; dz$$

which once more correctly encodes the range with the pole at $z=0$ disappearing when $q > m+1$. Therefore we may extend the range to n to get

$$\frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \sum_{q=0}^{n} \binom{n}{q} (-1)^q 2^q \frac{z^q}{(1+z)^q} \; dz$$
$$= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \left(1 - 2\frac{z}{1+z}\right)^n \; dz$$
$$= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{m+2}} \frac{(1-z)^n}{(1+z)^n} \; dz$$
$$= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1-z)^n}{z^{m+2}} \; dz$$
$$= (-1)^{m+1}\binom{n}{m+1}(-1)^{m+1} = \binom{n}{m+1}.$$

This was math.stackexchange.com problem 390321.

26 A triple Fibonacci-binomial coefficient convolution (B_1)

We seek to show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{k+1} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n-k} F_{2k+1}.$$

Start from
$$\binom{n+k}{n} = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^{n+1}} (1+z)^{n+k} \; dz.$$

This yields the following expression for the sum on the LHS

$$\frac{1}{2\pi i}\int_{|z|=1} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{z^{n+1}} (1+z)^{n+k} \frac{\varphi^{k+1} - (-1/\varphi)^{k+1}}{\sqrt{5}} \; dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1} \frac{(1+z)^n}{z^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \left(\varphi\left(\varphi(1+z)\right)^k + \frac{1}{\varphi}\left(-\frac{1}{\varphi}(1+z)\right)^k\right) \; dz$$

which finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(1+\varphi(1+z)\right)^n+\frac{1}{\varphi}\left(1-\frac{1}{\varphi}(1+z)\right)^n\right)\,dz$$

Continuing we have the following expression for the sum on the RHS

$$\frac{1}{2\pi i}\int_{|z|=1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{1}{z^{n+1}}(1+z)^{n+k}\frac{\varphi^{2k+1}-(-1/\varphi)^{2k+1}}{\sqrt{5}}\,dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\varphi\left(\varphi^2(1+z)\right)^k+\frac{1}{\varphi}\left(\frac{1}{\varphi^2}(1+z)\right)^k\right)\,dz$$

which finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(-1+\varphi^2(1+z)\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(1+z)\right)^n\right)\,dz$$

Apply the substitution $z=1/w$ to this integral to obtain (the sign to correct the reverse orientation of the circle is canceled by the minus on the derivative)

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n w^{n+1}$$
$$\times\left(\varphi\left(-1+\varphi^2(1+\frac{1}{w})\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(1+\frac{1}{w})\right)^n\right)\frac{1}{w^2}\,dw$$

which is

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n\frac{1}{w}$$
$$\times\left(\varphi\left(-w+\varphi^2(w+1)\right)^n+\frac{1}{\varphi}\left(-w+\frac{1}{\varphi^2}(w+1)\right)^n\right)\,dw$$

which finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\frac{(1+w)^n}{w^{n+1}}$$
$$\times\left(\varphi\left(-w+\varphi^2(w+1)\right)^n+\frac{1}{\varphi}\left(-w+\frac{1}{\varphi^2}(w+1)\right)^n\right)\,dw$$

This shows that the LHS is the same as the RHS because

$$-w+\varphi^2(w+1)=-w+(1+\varphi)(w+1)=1+\varphi(w+1)$$

and

$$-w + \frac{1}{\varphi^2}(w+1) = -w + (1-\frac{1}{\varphi})(w+1)$$
$$= -w + (w+1) - \frac{1}{\varphi}(w+1) = 1 - \frac{1}{\varphi}(w+1).$$

This is math.stackexchange.com problem 53830.

27 Fibonacci numbers and the residue at infinity (B_2R)

Suppose we seek to evaluate in terms of Fibonacci numbers

$$\sum_{p,q\ge 0} \binom{n-p}{q}\binom{n-q}{p}.$$

We use the integrals

$$\binom{n-p}{q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)^{q+1} z^{n-p-q+1}} \; dz$$

and

$$\binom{n-q}{p} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{(1-w)^{p+1} w^{n-p-q+1}} \; dw.$$

These correctly control the range so we may let p and q go to infinity to get for the sum (note that we get zero from the binomial coefficients when the upper index goes negative which is necessary for the sum to be finite, in general these vanish when the upper index is less than the lower one)

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{(1-w)w^{n+1}} \sum_{p,q\ge 0} \frac{z^{p+q}w^{p+q}}{(1-w)^p(1-z)^q} \; dw \; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{(1-w)w^{n+1}}$$
$$\times \frac{1}{1-zw/(1-w)} \frac{1}{1-zw/(1-z)} \; dw \; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w-zw} \frac{1}{1-z-zw} \; dw \; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{w-1/(1+z)} \frac{1}{w-(1-z)/z} \; dw \; dz.$$

Here we require for convergence that $|zw/(1-w)/(1-z)| < 1$. We have $|z/(1-z)| < \epsilon/(1-\epsilon)$ so we may take $\epsilon < 1/2$ and $\gamma < 1/2$ as well.

We evaluate the inner integral using the fact that the residues of the function in w sum to zero. We have two simple poles. We get for the first pole at $w = (1-z)/z = 1/z - 1$ that it lies on a circle of radius $1/\epsilon > 2$ centered at -1. This circle encloses the circle $|w| = \gamma < 1/2$ so this pole is outside the contour in w. For the second pole at $w = 1/(1+z)$ we get $|1/(1+z)| > 1/(1+\epsilon) > \epsilon$ which holds when $\epsilon < 1/2$ so this pole also is outside the contour. Collecting everything we have that we can evaluate the integral using minus the contribution from the residues of both poles.

Starting with the pole at $w = (1-z)/z$ we find

$$\frac{z^{n+1}}{(1-z)^{n+1}} \frac{1}{(1-z)/z - 1/(1+z)} = \frac{z^{n+1}}{(1-z)^{n+1}} \frac{z(1+z)}{(1-z)(1+z) - z}$$
$$= \frac{z^{n+2}}{(1-z)^{n+1}} \frac{1+z}{1-z-z^2}.$$

Substituting this expression into the outer integral we see that the pole at $z = 0$ is canceled making for a contribution of zero. The poles from the quadratic are outside the contour in z, as is the pole at $z = 1$. For the second pole at $w = 1/(1+z)$ we get

$$(1+z)^{n+1} \frac{1}{1/(1+z) - (1-z)/z} = (1+z)^{n+1} \frac{z(1+z)}{z - (1-z)(1+z)}.$$

This yields the contribution (taking into account the sign flip from the sum of residues)

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} (1+z)^{n+1} \frac{z(1+z)}{1-z-z^2} \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{1-z-z^2} \; dz.$$

We evaluate this using again the fact that the residues sum to zero. There are simple poles at $z = -\varphi$ and $z = 1/\varphi$ which are outside the contour in z.

These yield

$$\left(\frac{1-\varphi}{-\varphi}\right)^{n+1} \frac{1}{-1+2\varphi} + \left(\frac{1+1/\varphi}{1/\varphi}\right)^{n+1} \frac{1}{-1-2/\varphi}$$
$$= \frac{1}{\sqrt{5}} \frac{1}{\varphi^{2n+2}} - \frac{1}{\sqrt{5}} \varphi^{2n+2}.$$

Taking into account the sign flip this is obviously Binet / de Moivre for

$$F_{2n+2}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity in both cases is zero. For example in the first application we use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

which in the present case gives for the inner term in w

$$\begin{aligned} &-\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n+1} \frac{1}{1/w - 1/(1+z)} \frac{1}{1/w - (1-z)/z} \\ &= -\mathrm{Res}_{w=0} w^{n+1} \frac{1}{1 - w/(1+z)} \frac{1}{1 - w(1-z)/z} \end{aligned}$$

which is zero by inspection. Alternatively observe that we have a rational function with denominator degree exceeding numerator degree by at least two. The two poles in z do not vanish.

This was math.stackexchange.com problem 801730.

28 Permutations containing a given subsequence (B_1)

We seek to show that

$$S = \sum_{r=0}^{n} \binom{r+n-1}{n-1} \binom{3n-r}{n} = \frac{1}{2} \left\{ \binom{4n}{2n} + \binom{2n}{n}^2 \right\}.$$

This gives 1 on both sides when $n = 0$, we will work with $n \ge 1$. We write this as

$$S = S_2 - S_1 = \sum_{r=0}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n} - \sum_{r=n+1}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n}.$$

We start with S_2 and introduce

$$\binom{3n-r}{n} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{2n-r+1}} \frac{1}{(1-z)^{n+1}} \; dz$$

where $\varepsilon \ll 1$. Note that this will vanish when $r > 2n$ so we have for S_2

$$\begin{aligned} &\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{2n+1}} \frac{1}{(1-z)^{n+1}} \sum_{r\ge 0} \binom{r+n-1}{n-1} z^r \; dz \\ &= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{2n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{(1-z)^n} \; dz \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{2n+1}}\frac{1}{(1-z)^{2n+1}}\; dz = \binom{4n}{2n}.$$

This was the easy part. We now get for S_1

$$\sum_{r=0}^{n-1} \binom{r+2n}{n-1}\binom{2n-1-r}{n}.$$

Using the integral

$$\binom{2n-1-r}{n-1-r} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n-r}}(1+z)^{2n-1-r}\; dz$$

we see that this will vanish when $r \ge n$ so it controls the upper range of the sum and we obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n}}(1+z)^{2n-1}\sum_{r\ge 0}\binom{r+2n}{n-1}\frac{z^r}{(1+z)^r}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n}}(1+z)^{2n-1}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}(1+w)^{2n}\sum_{r\ge 0}(1+w)^r\frac{z^r}{(1+z)^r}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n}}(1+z)^{2n-1}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}(1+w)^{2n}\frac{1}{1-z(1+w)/(1+z)}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n}}(1+z)^{2n}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}(1+w)^{2n}\frac{1}{1-wz}\; dw\; dz.$$

Here we require for convergence that $|z(1+w)/(1+z)| < 1$. We have the upper bound $\varepsilon(1+\gamma)/(1-\varepsilon)$. Taking $\gamma = \varepsilon$ we require $2\varepsilon + \varepsilon^2 < 1$ so we may use $\varepsilon < 1/3$. Note that the pole at $w = 1/z$ is outside the unit circle and hence the contour and does not contribute, so we directly extract the residue in w to get

$$\sum_{q=0}^{n-1}\binom{2n}{n-1-q} z^q.$$

With the integral in z this will produce

$$\sum_{q=0}^{n-1}\binom{2n}{n-1-q}\binom{2n}{n-1-q} = \sum_{q=0}^{n-1}\binom{2n}{q}^2.$$

Now observe that

$$\sum_{q=n+1}^{2n}\binom{2n}{q}^2 = \sum_{q=0}^{n-1}\binom{2n}{2n-q}^2 = \sum_{q=0}^{n-1}\binom{2n}{q}^2.$$

We have by Vandermonde

$$\sum_{q=0}^{2n} \binom{2n}{q} \binom{2n}{2n-q} = \binom{4n}{2n}.$$

It follows that

$$\sum_{q=0}^{n-1} \binom{2n}{q}^2 = \frac{1}{2} \left\{ \binom{4n}{2n} - \binom{2n}{n}^2 \right\}.$$

That was S_1. Subtract from S_2 to get

$$\frac{1}{2} \left\{ \binom{4n}{2n} + \binom{2n}{n}^2 \right\}$$

as claimed.

This was math.stackexchange.com problem 1255356.

29 A binomial coefficient - Catalan number convolution (B_1)

A problem from MSE asks us to show that

$$\sum_{r=1}^{n+1} \frac{1}{r+1} \binom{2r}{r} \binom{m+n-2r}{n+1-r} = \binom{m+n}{n}.$$

This identity has a certain simplicity but it will be easier to include the term at $r=0$ where we need to show that

$$\sum_{r=0}^{n+1} \frac{1}{r+1} \binom{2r}{r} \binom{m+n-2r}{n+1-r} = \binom{m+n}{n} + \binom{m+n}{n+1} = \binom{m+n+1}{n+1}.$$

Here we may now take advantage of the fact that the second binomial coefficient enforces the upper range, so we obtain

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+2}} (1+w)^{m+n} \sum_{r\ge 0} \frac{1}{r+1} \binom{2r}{r} \frac{w^r}{(1+w)^{2r}} \; dw.$$

The Catalan number OGF is

$$\frac{1-\sqrt{1-4z}}{2z}$$

The binomial term $(1-4z)^{1/2}$ has radius of convergence $1/4$ about zero where we get series coefficient $(-1)^r 4^r \binom{1/2}{r}$ hence to close the sum we require

$|w/(1+w)^2| < 1/4$. (Note that there is no pole at $z=0$.) We get the upper bound $\gamma/(1-\gamma)^2$ so we may set $\gamma < 1/6$. We then obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+2}}(1+w)^{m+n}\frac{1-\sqrt{1-4w/(1+w)^2}}{2w/(1+w)^2}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+2}}(1+w)^{m+n+1}\frac{1+w-\sqrt{(1+w)^2-4w}}{2w}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+2}}(1+w)^{m+n+1}\frac{1+w-(1-w)}{2w}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+2}}(1+w)^{m+n+1}\; dw = \binom{m+n+1}{n+1}.$$

This is the claim.

This was math.stackexchange.com problem 563307.

30 A new obstacle from Concrete Mathematics (Catalan numbers) (B_1)

Suppose we seek to evaluate

$$\sum_{k\ge 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

where $m, n \ge 0$. In fact we may assume that $n \ge m$ because if $m > n$ the binomial coefficient vanishes using the definition $\binom{n}{k} = n^{\underline{k}}/k!$ because zero appears in the falling factorial.

Furthermore observe that when $k = n-m+q$ with $q > 0$ we obtain $\binom{2n-m+q}{2n-m+2q}$ which is zero by the same argument.

This gives

$$\sum_{k=0}^{n-m}\binom{n+k}{n-m-k}\binom{2k}{k}\frac{(-1)^k}{k+1}.$$

Introduce

$$\binom{n+k}{n-m-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-m-k+1}}(1+z)^{n+k}\; dz.$$

Observe that this is zero when $k > n-m$ so we may extend k to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-m+1}}(1+z)^n\sum_{k\ge 0}\binom{2k}{k}\frac{(-1)^k}{k+1}z^k(1+z)^k\; dz.$$

Here we recognize the generating function of the Catalan numbers

$$\sum_{k\ge 0} \binom{2k}{k} \frac{1}{k+1} w^k = \frac{1-\sqrt{1-4w}}{2w}$$

where the binomial $(1-4w)^{1/2}$ has coefficient $(-1)^k 4^k \binom{1/2}{k}$ and radius of convergence 1/4.We certainly have analyticity in a neighborhood of zero. We obtain

$$-\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} (1+z)^n \frac{1-\sqrt{1+4z(1+z)}}{2z(1+z)} \; dz$$
$$= -\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+2}} (1+z)^{n-1} \left(1-\sqrt{(1+2z)^2}\right) \; dz.$$

Now with z in a neighborhood of zero the square root produces the positive root so we finally have

$$-\frac{1}{2} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+2}} (1+z)^{n-1} (-2z) \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} (1+z)^{n-1} \; dz$$

which evaluates by inspection to $\binom{n-1}{n-m}$ which is

$$\binom{n-1}{m-1}.$$

Note that for the series to converge we need $|z(1+z)| < 1/4$ Nopw $|z(1+z)| \le \epsilon(1+\epsilon)$. With $\epsilon \ll 1$ this is less than 2ϵ. Therefor $\epsilon = 1/8$ will work.

This problem has not yet appeared at math.stackexchange.com.

31 Abel-Aigner identity from Table 202 of Concrete Mathematics (B_1)

Seeking to prove that

$$\sum_k \binom{tk+r}{k} \binom{tn-tk+s}{n-k} \frac{r}{tk+r} = \binom{tn+r+s}{n}$$

We will prove it with integers $t, r, s \ge 1$ and $r, s \ge t$. We then have it for arbitrary t, r, s because the sum is a polynomial in these variables and we have agreement on an infinite number of points. Under these additional asumptions we see that LHS of our identity is in fact

$$\sum_{k=0}^{n} \binom{tk+r}{k} \binom{tn-tk+s}{n-k} - \sum_{k=0}^{n} \binom{tk+r}{k} \binom{tn-tk+s}{n-k} \frac{tk}{tk+r}.$$

First do the case $t \ge 2$. Starting with the first sum we introduce

$$\binom{tk+r}{k} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k+1}} (1+w)^{tk+r} \; dw.$$

Now put $w/(1+w)^t = v$ and introduce the inverse $f(v) = w$. We then have by Lagrange inversion that $[v^n]f(v) = \frac{1}{tn+1}\binom{tn+1}{n} = \frac{1}{n}\binom{tn}{n-1}$ with the coefficient on the index zero term being zero. For the radius of convergence we have with $n \ge t$

$$\begin{aligned} &\frac{n}{n+1} \binom{tn+t}{n} \binom{tn}{n-1}^{-1} \\ &= \frac{n}{n+1} \frac{(n-1)!}{n!} \prod_{q=0}^{n-1} (tn+t-q) \prod_{q=0}^{n-2} (tn-q)^{-1} \\ &= \frac{1}{n+1} \prod_{q=0}^{t-1} (tn+t-q) \prod_{q=n-t}^{n-2} (tn-q)^{-1} \\ &= \frac{n}{n+1} \prod_{q=0}^{t-1} (t+t/n-q/n) \prod_{q=0}^{t-2} (t-(n-2)/n+q/n)^{-1} \sim \frac{t^t}{(t-1)^{t-1}}. \end{aligned}$$

It follows that $f(v)$ is analytic in a neighborhood of the origin with radius of convergence $\rho = (t-1)^{t-1}/t^t$. With $w/(1+w)^t = w + \cdots$ a convergent series the image of $|w| = \gamma$ approximates a circle $|v| = \gamma$ as $\gamma \ll 1$. Therefore we may deform the image to $|v| = \gamma$ where we take $\gamma < \rho$. We obtain

$$\frac{1}{2\pi i} \int_{|v|=\gamma} \frac{1}{v^k} \frac{1}{f(v)} (1+f(v))^r f'(v) \; dv.$$

From the definition of $f(v)$,

$$\frac{1}{2\pi i} \int_{|v|=\gamma} \frac{1}{v^{k+1}} (1+f(v))^{r-t} f'(v) \; dv.$$

Now introduce

$$\binom{tn-tk+s}{n-k} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n-k+1}} (1+z)^{tn-tk+s} \; dz.$$

This will vanish when $k \ge n$ so we may extend the sum to infinity, getting

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}(1+z)^{tn+s}\sum_{k\ge 0}\frac{z^k}{(1+z)^{tk}}$$
$$\times\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v^{k+1}}(1+f(v))^{r-t}f'(v)\;dv\;dz.$$

Note that the terms involving $f(v)$ and $f'(v)$ are analytic on and inside the contour hence the product has a convergent Laurent series there with empty principal part. In the sum we are extracting all cofficients of this series, evaluating the corresponding power at $z/(1+z)^t$. Hence we obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}(1+z)^{tn+s}(1+f(z/(1+z)^t))^{r-t}f'(z/(1+z)^t)\;dz.$$

Note that here we require $|z/(1+z)^t|<\rho$. With $|z/(1+z)^t|\le\varepsilon/(1-\varepsilon)^t$ we may take $\varepsilon<\varepsilon^*$ where ε^* is the unique solution of $\varepsilon=\rho(1-\varepsilon)^t$ in the interval $[0,1]$.

Computing the derivative we find

$$\frac{f'(v)}{(1+f(v))^t}-\frac{tf(v)}{(1+f(v))^{t+1}}f'(v)=1$$

so that

$$f'(v)=\frac{(1+f(v))^t}{1-tf(v)/(1+f(v))}.$$

We get for the integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}(1+z)^{tn+s}(1+z)^{r-t}\frac{(1+z)^t}{1-tz/(1+z)}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}(1+z)^{tn+r+s+1}\frac{1}{1+(1-t)z}\;dz.$$

Continuing with the second sum we obtain

$$t\sum_{k=1}^{n}\binom{tk+r-1}{k-1}\binom{tn-tk+s}{n-k}$$
$$=t\sum_{k=0}^{n-1}\binom{tk+t+r-1}{k}\binom{t(n-1)-tk+s}{(n-1)-k}.$$

We can recycle the earlier computation and find

$$\frac{t}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n}}(1+z)^{t(n-1)+r+t-1+s+1}\frac{1}{1+(1-t)z}\;dz$$

$$= \frac{t}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^n} (1+z)^{tn+r+s} \frac{1}{1+(1-t)z} \; dz.$$

Now we have subtracting the second piece from the first that $(1+z)/z - t = (1+z-tz)/z$ and we get

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} (1+z)^{tn+r+s} \; dz = \binom{tn+r+s}{n}$$

which is the claim. For $t = 1$ we get the following two sums, first,

$$\sum_{k=0}^{n} \binom{k+r}{k} \binom{n-k+s}{n-k} = [z^n] \frac{1}{(1-z)^{r+1}} \frac{1}{(1-z)^{s+1}}$$

$$= [z^n] \frac{1}{(1-z)^{r+s+2}} = \binom{n+r+s+1}{n}.$$

Repeat for the second sum to get

$$\binom{n-1+r+t-1+s+1}{n-1} = \binom{n+r+s}{n-1}.$$

The difference gives

$$\binom{n+r+s}{n}$$

as required.

This was math.stackexchange.com problem 2814898.

32 Reducing the form of a double hypergeometric (B_1)

Suppose we seek to evaluate

$$S(n) = \sum_{q=0}^{n-2} \sum_{k=1}^{n} \binom{k+q}{k} \binom{2n-q-k-1}{n-k+1}.$$

which we re-write as

$$-\sum_{q=0}^{n-2} \binom{2n-q-1}{n+1} - \sum_{q=0}^{n-2} \binom{n+1+q}{n+1} + \sum_{q=0}^{n-2} \sum_{k=0}^{n+1} \binom{k+q}{k} \binom{2n-q-k-1}{n-k+1}.$$

Call these pieces up to sign from left to right S_1, S_2 and S_3.

The two pieces in front cancel the quantities introduced by extending k to include the values zero and $n+1$.

Evaluation of S_1.

Introduce

$$\binom{2n-q-1}{n+1} = \binom{2n-q-1}{n-q-2} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-1}}{z^{n-q-1}}\; dz.$$

This vanishes when $q > n-2$ so we may extend the sum to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \sum_{q\ge 0} \frac{z^q}{(1+z)^q}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \frac{1}{1-z/(1+z)}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n-1}}\; dz$$

$$= \binom{2n}{n-2}.$$

Evaluation of S_2.
Introduce

$$\binom{n+1+q}{n+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1+q}}{z^{n+2}}\; dz.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \sum_{q=0}^{n-2} (1+z)^q\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \frac{(1+z)^{n-1}-1}{1+z-1}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+3}} ((1+z)^{n-1}-1)\; dz$$

$$= \binom{2n}{n+2}.$$

A more efficient evaluation is to notice that when we re-index q as $n-2-q$ in S_2 we obtain

$$\sum_{q=0}^{n-2} \binom{n+1+n-2-q}{n+1} = \sum_{q=0}^{n-2} \binom{2n-q-1}{n+1}$$

which is S_1.

Evaluation of S_3.

Introduce

$$\binom{2n-q-k-1}{n-k+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-k-1}}{z^{n-k+2}}\; dz.$$

This effectively controls the range so we can let k go to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2}\sum_{k\ge 0} \binom{k+q}{q} \frac{z^k}{(1+z)^{q+k}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2} \frac{1}{(1+z)^q} \frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \sum_{q=0}^{n-2} \frac{1}{(1+z)^{q+1}} \frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \times (n-1) \times\; dz$$

$$= (n-1) \times \binom{2n}{n+1}.$$

Finally collecting the three contributions we obtain

$$(n-1)\times\binom{2n}{n+1} - 2\binom{2n}{n+2} = (n+2)\binom{2n}{n+2} - 2\binom{2n}{n+2}$$

$$= n \times \binom{2n}{n+2}.$$

This is math.stackexchange.com problem 129913.

33 Basic usage of the Iverson bracket ($B_1 I$)

Suppose we seek to evaluate

$$S(k,l) = \sum_{q=0}^{l} \binom{q+k}{k}\binom{l-q}{k}.$$

We start with the Iverson bracket valid for $q \ge 0$

$$[[0 \le q \le l]] = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z^q}{z^{l+1}} \frac{1}{1-z}\; dz$$

This gives for the sum

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^l}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\sum_{q\ge 0}\binom{q+k}{q}\frac{z^q}{(1+w)^q}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^l}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(1-z/(1+w))^{k+1}}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(1+w-z)^{k+1}}\;dw\;dz.$$

We evaluate the inner integral by taking the negative of the sum of the residues at $z=1$ and at $z=1+w$ and $z=\infty$. With ϵ and γ small the second pole is not inside the contour.

The negative of the residue at $z=1$ is

$$\frac{1}{w^{k+1}}$$

which when substituted into the outer integral yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{2k+2}}\;dw = \binom{l+k+1}{2k+1},$$

which is the formula we are trying to establish.

Next we prove that the residue at infinity is zero. This is given by

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{l+1}\frac{1}{1-1/z}\frac{1}{(1+w-1/z)^{k+1}} = -\mathrm{Res}_{z=0}z^l\frac{1}{z-1}\frac{z^{k+1}}{(z(1+w)-1)^{k+1}}$$

$$= -\frac{1}{(1+w)^{k+1}}\mathrm{Res}_{z=0}\frac{1}{z-1}\frac{z^{l+k+1}}{(z-1/(1+w))^{k+1}}.$$

This is zero by inspection, which leaves the residue at $z=1+w$. Write

$$\frac{(-1)^{k+1}}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{l+k+1}}{w^{k+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{l+1}}\frac{1}{1-z}\frac{1}{(z-(1+w))^{k+1}}\;dw\;dz.$$

We require the derivative

$$\frac{1}{k!}\left(\frac{1}{z^{l+1}}\frac{1}{1-z}\right)^{(k)} = \frac{1}{k!}\sum_{q=0}^{k}\binom{k}{q}(-1)^q\frac{(l+q)!}{l!\times z^{l+1+q}}\frac{(k-q)!}{(1-z)^{1+k-q}}$$

$$= \sum_{q=0}^{k}\binom{l+q}{q}(-1)^q\frac{1}{z^{l+1+q}}\frac{1}{(1-z)^{1+k-q}}.$$

Evaluate this at $z = 1 + w$ to get

$$\sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}}$$

and substitute into the outer integral to obtain

$$\frac{(-1)^{k+1}}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{(1+w)^{l+1+q}} \frac{1}{w^{1+k-q}} \; dw$$

$$= \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{k-q}}{w^{2k+2-q}} \; dw.$$

The inner term here is

$$[w^{2k+1-q}](1+w)^{k-q}.$$

But we have $2k + 1 - q \ge k + 1$ while $k - q \le k$ so these terms are zero, thus concluding the proof.

Simplified solution. As observed elsewhere this can be done without the Iverson bracket.

Introduce

$$\binom{l-q}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-q-k+1}} \frac{1}{(1-z)^{k+1}} \; dz.$$

This controls the range becoming zero when $q > l - k$ so we may extend q to infinity.

We obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \sum_{q\ge 0} \binom{q+k}{k} z^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \frac{1}{(1-z)^{k+1}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{2k+2}} \; dz.$$

This evaluates by inspection to

$$\binom{l-k+2k+1}{2k+1} = \binom{l+k+1}{2k+1}.$$

This was math.stackexchange.com problem.

34 Basic usage of the Iverson bracket II ($B_1 I$)

Suppose we seek to compute

$$S(n, m) = \sum_{k=0}^{n} k \binom{m+k}{m+1}.$$

Introduce

$$\binom{m+k}{m+1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+k} \; dz$$

as well as the Iverson bracket

$$[[0 \le k \le n]] = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^k}{w^{n+1}} \frac{1}{1-w} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^m \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \sum_{k\ge 0} k w^k (1+z)^k \; dw \; dz.$$

For this to converge we must have $|w(1+z)| < 1$. We get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^m \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{w(1+z)}{(1-w(1+z))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+1} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-w} \frac{1}{(1-w(1+z))^2} \; dw \; dz.$$

We evaluate the inner integral using the fact that the residues at the poles sum to zero. The residue at $w = 1$ produces

$$-\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+2}} (1+z)^{m+1} \frac{1}{(-z)^2} \; dz = -\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+4}} (1+z)^{m+1} \; dz = 0.$$

For the residue at $w = 1/(1+z)$ we re-write the inner integral to get

$$\frac{1}{(1+z)^2} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{1-w} \frac{1}{(w-1/(1+z))^2} \; dw.$$

We thus require

$$\left.\left(\frac{1}{w^n}\frac{1}{1-w}\right)'\right|_{w=1/(1+z)}$$

$$= \left.\left(\frac{-n}{w^{n+1}}\frac{1}{1-w} + \frac{1}{w^n}\frac{1}{(1-w)^2}\right)\right|_{w=1/(1+z)}$$

$$= -n(1+z)^{n+1}(1+z)/z + (1+z)^n(1+z)^2/z^2.$$

Substituting this into the outer integral and flipping signs we get two pieces which are

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m-1} n(1+z)^{n+2}/z \; dz$$

$$= \frac{n}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+3}}(1+z)^{n+m+1} \; dz = n \times \binom{n+m+1}{m+2}.$$

The second piece is

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m-1}(1+z)^{n+2}/z^2 \; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+4}}(1+z)^{n+m+1} \; dz = -\binom{n+m+1}{m+3}.$$

It follows that our answer is

$$\left(n - \frac{n-1}{m+3}\right)\binom{n+m+1}{m+2} = \frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

Remark. Being rigorous we also verify that the residue at infinity in the calculation of the inner integral is zero. We get

$$-\mathrm{Res}_{w=0}\frac{1}{w^2}w^n\frac{1}{1-1/w}\frac{1}{(1-(1+z)/w)^2}$$

$$= -\mathrm{Res}_{w=0} w^{n-2}\frac{w}{w-1}\frac{w^2}{(w-(1+z))^2} = -\mathrm{Res}_{w=0}\frac{w^{n+1}}{w-1}\frac{1}{(w-(1+z))^2}.$$

There is certainly no pole at zero here and the residue is zero as claimed (the term $1+z$ rotates in a circle around the point one on the real axis and with $\epsilon < 1$ it is never zero). This last result could also be obtained by comparing degrees of numerator and denominator.

This was math.stackexchange.com problem 1836190.

35 Iverson bracket used twice (B_1IR)

Suppose we seek to evaluate

$$Y(n) = \sum_{k=0}^{n} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor},$$

by considering

$$Y_1(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{2k}{k} \quad \text{and} \quad Y_2(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{n-2k-1} \binom{2k+1}{k}.$$

We will use the following Iverson bracket:

$$[[0 \le k \le n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz$$

where we must have $\epsilon \ll 1$.

Evaluation of $Y_1(n)$.

Introduce

$$\binom{2k}{k} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k+1}} (1+w)^{2k} \; dw.$$

With the Iverson bracket controlling the range we can extend k to infinity to get for the sum

$$\frac{2^n}{2\pi i} \int_{|w|=\gamma} \frac{1}{w} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{1-z} \sum_{k \ge 0} 2^{-2k} z^k \frac{(1+w)^{2k}}{w^k} \; dz \; dw.$$

We can instantiate these contours to get convergence of the series. In particular we require $|z(1+w)^2/w/4| < 1$. We have $|z(1+w)^2/w/4| \le \epsilon(1+\gamma)^2/\gamma/4$ hence we may take $\epsilon = \gamma$ and $\gamma \ll 1$.

We thus obtain

$$\frac{2^n}{2\pi i} \int_{|w|=\gamma} \frac{1}{w} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{1-z} \frac{1}{1-z(1+w)^2/w/4} \; dz \; dw$$

$$= \frac{2^{n+2}}{2\pi i} \int_{|w|=\gamma} \frac{1}{(1+w)^2} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2 \rfloor + 1}} \frac{1}{z-1} \frac{1}{z - 4w/(1+w)^2} \; dz \; dw.$$

We evaluate the inner piece by computing the negative of the sum of the residues at $z = 1$, $z = 4w/(1+w)^2$ and $z = \infty$. This works because $z = 1$ and $z = \infty$ are outside of the contour as is $z = 4w/(1+w)^2$ because $|z| < |4w/(1+w)^2|$ by the constraint on the convergence.

We get for $z = 1$

$$\frac{1}{1-4w/(1+w)^2} = \frac{(1+w)^2}{(1+w)^2-4w} = \frac{(1+w)^2}{(1-w)^2}$$

for a zero contribution.

We get for $z = \infty$

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{1}{1/z^{\lfloor n/2 \rfloor+1}} \frac{1}{1/z-1} \frac{1}{1/z-4w/(1+w)^2}$$

$$= -\mathrm{Res}_{z=0} z^{\lfloor n/2 \rfloor+1} \frac{1}{1-z} \frac{1}{1-4wz/(1+w)^2}$$

again for a zero contribution.

Finally for $z = 4w/(1+w)^2$ we get

$$-\frac{(1+w)^{2\lfloor n/2 \rfloor+2}}{2^{2\lfloor n/2 \rfloor+2} \times w^{\lfloor n/2 \rfloor+1}} \frac{(1+w)^2}{(1-w)^2}.$$

Substitute into the outer integral to obtain

$$-\frac{2^{n \mod 2}}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{2\lfloor n/2 \rfloor+2}}{w^{\lfloor n/2 \rfloor+1}} \frac{1}{(1-w)^2} \; dw.$$

Extracting the negative of the residue we get the sum

$$2^{n \mod 2} \sum_{q=0}^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor+2}{q} (\lfloor n/2 \rfloor - q + 1).$$

This yields

$$2^{n \mod 2} (\lfloor n/2 \rfloor + 1) \frac{1}{2} \left(2^{2\lfloor n/2 \rfloor+2} - \binom{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right)$$

$$-2^{n \mod 2} (2\lfloor n/2 \rfloor + 2) \sum_{q=1}^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor+1}{q-1}$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor + 1) \frac{1}{2} \left(2^{2\lfloor n/2 \rfloor+2} - \binom{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right)$$

$$-2^{n \mod 2} (\lfloor n/2 \rfloor + 1) \left(2^{2\lfloor n/2 \rfloor+1} - 2\binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor} \right)$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor + 1) \left(2 - \frac{1}{2} \frac{2\lfloor n/2 \rfloor+2}{\lfloor n/2 \rfloor+1} \right) \binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor}$$

$$= 2^{n \mod 2} (\lfloor n/2 \rfloor + 1) \binom{2\lfloor n/2 \rfloor+1}{\lfloor n/2 \rfloor}.$$

Evaluation of $Y_2(n)$.
We get

$$Y_2(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{n-2k-2} \binom{2k+2}{k+1} = \sum_{k=1}^{1+\lfloor (n-1)/2 \rfloor} 2^{n-2k} \binom{2k}{k} = -2^n + \frac{1}{2} Y_1(n+1).$$

Evaluation of $Y(n)$.
We get for $n = 2p$ the contributions

$$\begin{aligned} -2^{2p}(p+1)\binom{2p+1}{p} - 2^{2p} + (p+1)\binom{2p+1}{p} \\ = -2^{2p} + (4p+2)\binom{2p}{p}. \end{aligned}$$

On the other hand for $n = 2p+1$ we obtain

$$\begin{aligned} 2(p+1)\binom{2p+1}{p} - 2^{2p+1} + \frac{1}{2}(p+2)\binom{2p+3}{p+1} \\ = -2^{2p+1} + (4p+5)\binom{2p+1}{p}. \end{aligned}$$

Joining the two formulae we get the compact closed form

$$-2^n + (2n + 2 + (n \mod 2))\binom{n}{\lfloor n/2 \rfloor}.$$

Alternate proof.
Using

$$\sum_{k \ge 0} \binom{2k}{k} z^{2k} = \frac{1}{\sqrt{1-4z^2}}$$

and

$$\sum_{k \ge 0} \binom{2k+1}{k} z^{2k+1} = \frac{1}{2z} \sum_{k \ge 0} \binom{2k+2}{k+1} z^{2k+2} = -\frac{1}{2z} + \frac{1}{2z} \frac{1}{\sqrt{1-4z^2}}$$

we obtain

$$\begin{aligned} Y(n) &= [z^n] \frac{1}{1-2z} \left[-\frac{1}{2z} + \frac{1}{2z} \frac{1}{\sqrt{1-4z^2}} + \frac{1}{\sqrt{1-4z^2}} \right] \\ &= -2^n + [z^n] \frac{1}{1-2z} \left[\frac{1}{2z} \frac{1}{\sqrt{1-4z^2}} + \frac{1}{\sqrt{1-4z^2}} \right]. \end{aligned}$$

Now observe that

$$[z^{2p}]\frac{1}{1-2z}\frac{1}{\sqrt{1-4z^2}} = \frac{1}{2}[z^{2p}]\frac{2}{(1-4z^2)^{3/2}} = \binom{-3/2}{p}(-1)^p 4^p$$
$$= \binom{-1/2}{p}(-1)^p 4^p \frac{-3/2-(p-1)}{-1/2} = (2p+1)\binom{2p}{p}$$

and

$$[z^{2p+1}]\frac{1}{1-2z}\frac{1}{\sqrt{1-4z^2}} = \frac{1}{2}[z^{2p+1}]\frac{4z}{(1-4z^2)^{3/2}} = 2(2p+1)\binom{2p}{p}.$$

Collecting everything we have for $n = 2p$

$$-2^{2p} + (2p+1)\binom{2p}{p} + (2p+1)\binom{2p}{p} = -2^{2p} + (4p+2)\binom{2p}{p}$$

and for $n = 2p+1$

$$-2^{2p+1} + 2(2p+1)\binom{2p}{p} + \frac{1}{2}(2p+3)\binom{2p+2}{p+1}$$
$$= -2^{2p+1} + 2(p+1)\binom{2p+1}{p} + (2p+3)\binom{2p+1}{p}$$
$$= -2^{2p+1} + (4p+5)\binom{2p+1}{p}.$$

This is the same result as what we obtained earlier.
This was math.stackexchange.com problem 1219731.

36 Iverson bracket and an identity by Gosper, generalized (IR)

Suppose we seek to show that

$$\sum_{q=0}^{m-1}\binom{n-1+q}{q}x^n(1-x)^q + \sum_{q=0}^{n-1}\binom{m-1+q}{q}x^q(1-x)^m = 1$$

where $n, m \geq 1$.

We will evaluate the second term by a contour integral and show that is equal to one minus the first term which is the desired result.

Introduce the Iverson bracket

$$[[0 \leq q \leq n-1]] = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{z^q}{z^n}\frac{1}{1-z}\,dz.$$

With this bracket we may extend the sum in q to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\sum_{q\ge 0}\binom{m-1+q}{q} z^q x^q (1-x)^m \; dz$$

$$= \frac{(1-x)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\sum_{q\ge 0}\binom{m-1+q}{q} z^q x^q \; dz$$

$$= \frac{(1-x)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\frac{1}{(1-xz)^m} \; dz.$$

For the series to converge we need $|xz| < 1$. With $|z| = \epsilon \ll 1$ we may restrict to $|x| < 1$. The claim then holds for all x because the sum is a polynomial in x. Now we have three poles here at $z = 0$ and $z = 1$ and $z = 1/x$ and the residues at these poles sum to zero, so we can evaluate the residue at zero by computing the negative of the residues at $z = 1$ and $z = 1/x$.

Observe that the residue at infinity is zero as can be seen from the following computation:

$$-\mathrm{Res}_{z=0}\frac{1}{z^2} z^n \frac{1}{1-1/z}\frac{1}{(1-x/z)^m}$$

$$-\mathrm{Res}_{z=0}\frac{1}{z^2} z^n \frac{z}{z-1}\frac{z^m}{(z-x)^m}$$

$$-\mathrm{Res}_{z=0} z^{n+m-1}\frac{1}{z-1}\frac{1}{(z-x)^m} = 0.$$

Returning to the main thread the residue at $z = 1$ as seen from

$$-\frac{(1-x)^m}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{z-1}\frac{1}{(1-xz)^m} \; dz.$$

is

$$-(1-x)^m \frac{1}{(1-x)^m} = -1.$$

For the residue at $z = 1/x$ we consider

$$\frac{(1-x)^m}{x^m \times 2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\frac{1}{(1/x-z)^m} \; dz$$

$$= \frac{(-1)^m(1-x)^m}{x^m \times 2\pi i}\int_{|z|=\epsilon}\frac{1}{z^n}\frac{1}{1-z}\frac{1}{(z-1/x)^m} \; dz.$$

and use the following derivative:

$$\frac{1}{(m-1)!}\left(\frac{1}{z^n}\frac{1}{1-z}\right)^{(m-1)}$$

$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1)^q (n+q-1)!}{(n-1)! z^{n+q}} \frac{(m-1-q)!}{(1-z)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \frac{1}{q!} \frac{(-1)^q (n+q-1)!}{(n-1)! z^{n+q}} \frac{1}{(1-z)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \binom{n+q-1}{q} \frac{(-1)^q}{z^{n+q}} \frac{1}{(1-z)^{m-q}}.$$

Evaluate this at $z = 1/x$ and multiply by the factor in front to get

$$\frac{(-1)^m (1-x)^m}{x^m} \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^{n+q} \frac{1}{(1-1/x)^{m-q}}$$

$$= \frac{(-1)^m (1-x)^m}{x^m} \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^{n+q} \frac{x^{m-q}}{(x-1)^{m-q}}$$

$$= (-1)^m (1-x)^m \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^n (-1)^{m-q} \frac{1}{(1-x)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \binom{n+q-1}{q} x^n (1-x)^q.$$

This yields for the second sum term the value

$$1 - \sum_{q=0}^{m-1} \binom{n+q-1}{q} x^n (1-x)^q$$

showing that when we add the first and the second sum by cancellation the end result is one, as claimed.

This was math.stackexchange.com problem 538309.

Special case by formal power series

Here we show the special case:

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} + \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k} = 2^{n+m+1}.$$

which is obtained from $x = 1/2$. We have by inspection i.e. same as before that

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} = 2^n [z^n] \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}}.$$

This is

$$
\begin{aligned}
&2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}} \\
&= -2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{2^{m+1}}{(2-z)^{m+1}} \\
&= 2^{n+m+1}(-1)^m \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}.
\end{aligned}
$$

With

$$f(z) = 2^{n+m+1}(-1)^m \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}$$

we will be using the fact that residues sum to zero i.e.

$$\mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=1} f(z) + \mathrm{Res}_{z=2} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

The residue at infinity is zero since $\lim_{R\to\infty} 2\pi R/R^{n+1}/R/R^{m+1} = 0$. The residue at one is

$$2^{n+m+1}(-1)^m \times (-1)^{m+1} = -2^{n+m+1}.$$

For the residue at two we use the Leibniz rule:

$$
\begin{aligned}
&\frac{1}{m!}\left(\frac{1}{z^{n+1}} \frac{1}{z-1}\right)^{(m)} \\
&= \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{(n+k)!}{n!} \frac{1}{z^{n+1+k}} (-1)^{m-k} \frac{(m-k)!}{(z-1)^{m-k+1}} \\
&= (-1)^m \sum_{k=0}^{m} \binom{n+k}{k} \frac{1}{z^{n+1+k}} \frac{1}{(z-1)^{m-k+1}}.
\end{aligned}
$$

Restore factor in front and evaluate at $z = 2$:

$$2^{n+m+1}(-1)^m \times (-1)^m \sum_{k=0}^{m} \binom{n+k}{k} \frac{1}{2^{n+1+k}} = \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k}.$$

Summing the residues we have shown that

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} + \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k} - 2^{n+m+1} = 0$$

which is the claim.
This was math.stackexchange.com problem 3024722.

37 Factoring a triple hypergeometric sum (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n}(-1)^k \binom{p+q+1}{k}\binom{p+n-k}{n-k}\binom{q+n-k}{n-k}$$

which is claimed to be

$$\binom{p}{n}\binom{q}{n}.$$

We will prove it for $p, q \geq n$ and then we have it for all p, q because the sum is a polynomial in p, q.

Re-write as follows:

$$(-1)^n \sum_{k=0}^{n}(-1)^k \binom{p+q+1}{n-k}\binom{p+k}{k}\binom{q+k}{k}.$$

Introduce

$$\binom{p+q+1}{n-k} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n-k+1}}(1+z)^{p+q+1}\,dz$$

and

$$\binom{q+k}{q} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}}(1+w)^{q+k}\,dw$$

Note that the first integral vanishes when $k > n$ so we may extend k to infinity, getting

$$(-1)^n \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^{p+q+1} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}}(1+w)^{q}$$
$$\times \sum_{k\geq 0}\binom{p+k}{k}(-1)^k z^k (1+w)^k\,dw\,dz.$$

Here we have $\varepsilon < 1$ and $\gamma < 1$ and we require for convergence that $|z(1+w)| < 1$. We have $|z(1+w)| \leq \varepsilon(1+\gamma)$ so we may choose $\varepsilon = \gamma$ and $\gamma < 1/2$. Continuing,

$$(-1)^n \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^{p+q+1} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}}(1+w)^{q}$$
$$\times \frac{1}{(1+z(1+w))^{p+1}}\,dw\,dz$$
$$= (-1)^n \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+p+2}}(1+z)^{p+q+1} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}}(1+w)^{q}$$

$$\times \frac{1}{(w + (1+z)/z)^{p+1}} \; dw \; dz.$$

Here the residue at infinity in w is zero by inspection and the pole at $w = -(1+z)/z$ has norm lower bounded by $(1-\gamma)/\gamma = 1/\gamma - 1 > \gamma$ so it is outside the contour. Hence we may evaluate using minus the residue at that pole with the Leibniz rule:

$$\begin{aligned} &\frac{1}{p!}\left(\frac{1}{w^{q+1}}(1+w)^q\right)^{(p)} \\ &= \frac{1}{p!}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k (q+1)^{\overline{k}}}{w^{q+1+k}} q^{\underline{p-k}}(1+w)^{q-(p-k)} \\ &= \sum_{k=0}^{p}(-1)^k\binom{q+k}{k}\frac{1}{w^{q+1+k}}\binom{q}{p-k}(1+w)^{q+k-p}. \end{aligned}$$

Evaluate at $w = -(1+z)/z$ and flip sign to get

$$-\sum_{k=0}^{p}(-1)^k\binom{q+k}{k}(-1)^{q+1+k}\frac{z^{q+1+k}}{(1+z)^{q+1+k}}\binom{q}{p-k}(-1)^{q+k-p}\frac{1}{z^{q+k-p}}$$

$$= (-1)^p\sum_{k=0}^{p}(-1)^k\binom{q+k}{k}\binom{q}{p-k}\frac{z^{p+1}}{(1+z)^{q+1+k}}.$$

Substitute into the integral to obtain

$$(-1)^{n+p}\sum_{k=0}^{p}(-1)^k\binom{q+k}{k}\binom{q}{p-k}\binom{p-k}{n}.$$

Now note that

$$\binom{q}{p-k}\binom{p-k}{n} = \frac{q!}{(q-(p-k))! \times n! \times (p-k-n)!} = \binom{q}{n}\binom{q-n}{p-k-n}.$$

Both sides here are zero when $q < p-k$ and we have used the initial condition $q \ge n$ in this step. We obtain

$$(-1)^{n+p}\binom{q}{n}\sum_{k=0}^{p}(-1)^k\binom{q+k}{k}\binom{q-n}{p-k-n}.$$

We have obtained the first factor. With the standard extractor on the remaining sum using the initial condition $p \ge n$

$$(-1)^{n+p}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{p-n+1}}(1+z)^{q-n}\sum_{k\ge 0}(-1)^k\binom{q+k}{k}z^k \; dz.$$

Here the residue enforces $k \le p-n$ so we may extend to infinity, with $\varepsilon < 1$. We find

$$(-1)^{n+p}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{p-n+1}}(1+z)^{-n-1}\;dz = (-1)^{n+p}\binom{-n-1}{p-n} = \binom{p}{p-n} = \binom{p}{n}.$$

We have obtained the second factor and may conclude.

This is math.stackexchange.com problem 174054.

38 Factoring a triple hypergeometric sum II (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{pn-n}{k}\binom{pn+k}{k} = \binom{pn}{n}^2.$$

We use the integrals

$$\binom{pn-n}{k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{pn-n}}{z^{k+1}}\;dz$$

and

$$\binom{pn+k}{k} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pn+k}}{w^{k+1}}\;dw.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{pn-n}}{z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pn}}{w}\sum_{k=0}^{n}\binom{n}{k}\frac{(1+w)^k}{z^k w^k}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{pn-n}}{z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pn}}{w}\left(1+\frac{1+w}{zw}\right)^n\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{pn-n}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pn}}{w^{n+1}}(1+w+zw)^n\;dw\;dz.$$

Expanding the binomial in the inner sum we get

$$\sum_{q=0}^{n}\binom{n}{q}w^q(1+z)^q$$

which yields

$$\sum_{q=0}^{n}\binom{n}{q}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{pn-n+q}}{z^{n+1}}\binom{pn}{n-q}\;dz$$

$$= \sum_{q=0}^{n} \binom{n}{q} \binom{pn-n+q}{n} \binom{pn}{n-q}.$$

The inner term is

$$\binom{n}{q} \binom{pn-n+q}{n} \binom{pn}{pn-n+q}$$

$$= \frac{(pn)!}{q! \times (n-q)! \times (pn-2n+q)! \times (n-q)!}$$

$$= \binom{pn}{n} \frac{n! \times (pn-n)!}{q! \times (n-q)! \times (pn-2n+q)! \times (n-q)!}$$

$$= \binom{pn}{n} \binom{n}{q} \binom{pn-n}{n-q}.$$

Thus it remains to show that

$$\sum_{q=0}^{n} \binom{n}{q} \binom{pn-n}{n-q} = \binom{pn}{n}.$$

This can be done combinatorially or using the integral

$$\frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn-n}}{v^{n+1}} \sum_{q=0}^{n} \binom{n}{q} v^q \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn-n}}{v^{n+1}} (v+1)^n \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{(1+v)^{pn}}{v^{n+1}} = \binom{pn}{n}.$$

This was math.stackexchange.com problem 656116.

39 Factoring a triple hypergeometric sum III (B_1)

Suppose we seek to verify that

$$\sum_{r=0}^{\min\{m,n,p\}} \binom{m}{r} \binom{n}{r} \binom{p+m+n-r}{m+n} = \binom{p+m}{m} \binom{p+n}{n}.$$

Introduce

$$\binom{n}{r} = \binom{n}{n-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-r+1}} (1+z)^n \; dz$$

and

$$\binom{p+m+n-r}{m+n} = \binom{p+m+n-r}{p-r}$$
$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p-r+1}}(1+w)^{p+m+n-r}\; dw.$$

Observe carefully that the first of these is zero when $r > n$ and the second one when $r > p$ so we may extend the range of r to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{p+m+n}}{w^{p+1}} \sum_{r\ge 0} \binom{m}{r} z^r \frac{w^r}{(1+w)^r}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{p+m+n}}{w^{p+1}} \left(1+\frac{zw}{1+w}\right)^m\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{p+n}}{w^{p+1}} (1+w+zw)^m\; dw\; dz.$$

The inner integral is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{p+n}}{w^{p+1}} \sum_{q=0}^{m} \binom{m}{q} (1+z)^q w^q\; dw$$

with residue

$$\sum_{q=0}^{\min(m,p)} \binom{m}{q}\binom{p+n}{p-q}(1+z)^q$$

which in combination with the outer integral yields

$$\sum_{q=0}^{\min(m,p)} \binom{m}{q}\binom{p+n}{n+q}\binom{n+q}{n}.$$

Now note that

$$\binom{p+n}{n+q}\binom{n+q}{n} = \frac{(p+n)!}{(p-q)!(n+q)!}\frac{(n+q)!}{q!n!}$$
$$= \frac{(p+n)!}{(p-q)!p!}\frac{p!}{q!n!} = \binom{p+n}{n}\binom{p}{q}.$$

Therefore we just need to verify that

$$\sum_{q=0}^{\min(m,p)} \binom{m}{q}\binom{p}{p-q} = \binom{p+m}{m}$$

which follows by inspection.
It can also be done with the integral

$$\binom{p}{p-q} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^p}{w^{p-q+1}}\; dw$$

which is zero when $q > p$ so we can extend q to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^p}{w^{p+1}} \sum_{q\ge 0} \binom{m}{q} w^q \; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{p+m}}{w^{p+1}}\; dw$$

$$= \binom{p+m}{m}.$$

This was math.stackexchange.com problem 1460712.

40 A triple hypergeometric sum IV (B_1)

Suppose we seek to verify that

$$\sum_{p=0}^{l}\sum_{q=0}^{p} (-1)^q \binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q} = 2^l \binom{m-n}{l}$$

where $m \ge n$ and $m-n \ge l$.
This is

$$\sum_{p=0}^{l} \binom{m-p}{m-l} \sum_{q=0}^{p} (-1)^q \binom{n}{q}\binom{m-n}{p-q}.$$

Now introduce the integral

$$\binom{m-n}{p-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{p-q+1}} (1+z)^{m-n}\; dz.$$

Note that this vanishes when $q > p$ so we may extend the range of q to infinity, getting for the sum

$$\sum_{p=0}^{l} \binom{m-p}{m-l} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{p+1}} (1+z)^{m-n} \sum_{q\ge 0} (-1)^q \binom{n}{q} z^q\; dz$$

$$= \sum_{p=0}^{l} \binom{m-p}{l-p} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{p+1}} (1+z)^{m-n} (1-z)^n\; dz.$$

Introduce furthermore

$$\binom{m-p}{l-p} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l-p+1}}(1+w)^{m-p}\; dw.$$

This too vanishes when $p > l$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l+1}}(1+w)^{m}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}(1+z)^{m-n}(1-z)^n \sum_{p\ge 0} \frac{w^p}{z^p}\frac{1}{(1+w)^p}\; dz\; dw.$$

The geometric series converges when $|w/z/(1+w)| < 1$. We get

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l+1}}(1+w)^{m}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}(1+z)^{m-n}(1-z)^n \frac{1}{1-w/z/(1+w)}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l+1}}(1+w)^{m}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} (1+z)^{m-n}(1-z)^n \frac{1}{z-w/(1+w)}\; dz\; dw.$$

Now from the convergence we have $|w/(1+w)| < |z|$ which means the pole at $z = w/(1+w)$ is inside the contour $|z| = \epsilon$. Extracting the residue yields (the pole at zero has disappeared)

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l+1}}(1+w)^{m}\left(1+\frac{w}{1+w}\right)^{m-n}\left(1-\frac{w}{1+w}\right)^{n}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{l+1}}(1+2w)^{m-n}\; dw$$

$$= 2^l\binom{m-n}{l}.$$

This was math.stackexchange.com problem 1767709.

41 Basic usage of exponentiation integral to obtain Stirling number formulae (E)

Suppose we seek to evaluate

$$\sum_{q=0}^{n} (n-2q)^k \binom{n}{2q+1}.$$

We observe that

$$(n-2q)^k = \frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k+1}} \exp((n-2q)z)\; dz.$$

This yields for the sum

$$\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k+1}} \sum_{q=0}^{n} \binom{n}{2q+1} \exp((n-2q)z)\; dz$$

$$= \frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}} \sum_{q=0}^{n} \binom{n}{2q+1} \exp((-2q-1)z)\; dz$$

which is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}}$$

$$\times \left(\sum_{q=0}^{n} \binom{n}{q} \exp(-qz) - \sum_{q=0}^{n} \binom{n}{q} (-1)^q \exp(-qz)\right)\; dz.$$

This yields two pieces, call them A_1 and A_2. Piece A_1 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}} (1+\exp(-z))^n\; dz$$

$$= \frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} (\exp(z)+1)^n\; dz$$

and piece A_2 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}} (1-\exp(-z))^n\; dz$$

$$= \frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} (\exp(z)-1)^n\; dz.$$

Recall the species equation for labelled set partitions:

$$\mathfrak{P}(\mathcal{U}\mathfrak{P}_{\ge 1}(\mathcal{Z}))$$

which yields the bivariate generating function of the Stirling numbers of the second kind

$$\exp(u(\exp(z)-1)).$$

This implies that

$$\sum_{n\ge q} \left\{ n \atop q \right\} \frac{z^n}{n!} = \frac{(\exp(z)-1)^q}{q!}$$

and

$$\sum_{n\ge q} \left\{ n \atop q \right\} \frac{z^{n-1}}{(n-1)!} = \frac{(\exp(z)-1)^{q-1}}{(q-1)!} \exp(z).$$

Now to evaluate A_1 proceed as follows:

$$\frac{1}{2} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} (2+\exp(z)-1)^n \; dz$$

$$= \frac{1}{2} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} \sum_{q=0}^{n} {n \choose q} 2^{n-q} (\exp(z)-1)^q \; dz$$

$$= \sum_{q=0}^{n} {n \choose q} 2^{n-q} \times q! \times \frac{1}{2} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} \frac{(\exp(z)-1)^q}{q!} \; dz.$$

Recognizing the differentiated Stirling number generating function this becomes

$$\sum_{q=0}^{n} {n \choose q} 2^{n-q-1} \times q! \times \left\{ k+1 \atop q+1 \right\}.$$

Now observe that when $n > k+1$ the Stirling number for $k+1 < q \le n$ is zero, so we may replace n by $k+1$. Similarly, when $n < k+1$ the binomial coefficient for $n < q \le k+1$ is zero so we may again replace n by $k+1$. This gives the following result for A_1 :

$$\sum_{q=0}^{k+1} {n \choose q} 2^{n-q-1} \times q! \times \left\{ k+1 \atop q+1 \right\}.$$

Moving on to A_2 we observe that when $k < n$ the contribution is zero because the series for $\exp(z)-1$ starts at z. This integral is simple and we have

$$\frac{1}{2} \frac{k! \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} \frac{(\exp(z)-1)^n}{n!} \; dz.$$

Recognizing the Stirling number this yields

$$\frac{1}{2} \times n! \times \left\{ k+1 \atop n+1 \right\}.$$

which correctly represents the fact that we have a zero contribution when $k < n$.

This finally yields the closed form formula

$$\sum_{q=0}^{k+1} {n \choose q} 2^{n-q-1} \times q! \times \left\{ k+1 \atop q+1 \right\} - \frac{1}{2} \times n! \times \left\{ k+1 \atop n+1 \right\}.$$

confirming the previous results.

This was math.stackexchange.com problem 1353963

42 Three phase application including Leibniz' rule (B_1B_2R)

Suppose we seek to verify that

$$\sum_{q=0}^{n} q \binom{2n}{n+q}\binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \geq m$.

We use the integrals

$$\binom{2n}{n+q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-q+1}} \frac{1}{(1-z)^{n+q+1}} \; dz.$$

and

$$\binom{m+q-1}{2m-1} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{m+q-1}}{w^{2m}} \; dw.$$

Observe that the first integral is zero when $q > n$ so we may extend q to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \sum_{q\geq 0} q \frac{z^q(1+w)^q}{(1-z)^q} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)/(1-z)}{(1-z(1+w)/(1-z))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)(1-z)}{(1-z-z(1+w))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n}} \frac{1}{(1-z)^{n}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{m}}{w^{2m}} \frac{1}{(1-2z-zw)^2} \; dw \; dz.$$

We evaluate the inner integral using the negative of the residue at the pole at $w = (1-2z)/z$, starting from

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}} \frac{1}{(1-z)^n} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^m}{w^{2m}} \frac{1}{(w-(1-2z)/z)^2} \; dw \; dz.$$

Differentiating we have

$$m\frac{(1+w)^{m-1}}{w^{2m}} - 2m\frac{(1+w)^m}{w^{2m+1}} = (w - 2(1+w))\, m\frac{(1+w)^{m-1}}{w^{2m+1}}$$

$$= (-w-2)m\frac{(1+w)^{m-1}}{w^{2m+1}}.$$

The negative of this evaluated at $w = (1-2z)/z$ is

$$\frac{1}{z} \times m \times \frac{(1-z)^{m-1}}{z^{m-1}} \times \frac{z^{2m+1}}{(1-2z)^{2m+1}}$$

which finally yields

$$\frac{m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}} \; dz.$$

We have that the residues at zero, one and one half sum to zero with the first one being the sum we are trying to compute. Therefore we evaluate these in turn. We will restore the front factor of m at the end.

For the residue at zero we have using the Cauchy product that

$$\sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{2m+n-m-q}{n-m-q}$$

$$= \sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{m+n-q}{2m}.$$

For the residue at one we have that

$$\frac{(-1)^{n-m+1}}{(n-m)!} \left(\frac{1}{z^{n-m+1}} \frac{1}{(1-2z)^{2m+1}} \right)^{(n-m)}$$

$$= \frac{(-1)^{n-m+1}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{n-m-q} \frac{(2m+n-m-q)!}{(2m)! \times (1-2z)^{2m+1+n-m-q}}$$

$$= \frac{(-1)^{n-m+1} 2^{n-m}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{-q}\frac{(m+n-q)!}{(2m)!\times(1-2z)^{m+1+n-q}}.$$

Evaluate this at one to get

$$2^{n-m}\sum_{q=0}^{n-m}\binom{n-m+q}{q}2^{-q}\binom{m+n-q}{2m}.$$

The residue at one evaluates to the sum we seek just like the residue at zero. This leaves the residue at one half, where we find

$$\frac{(-1)^{2m+1}}{(2m)!\times 2^{2m+1}}\left(\frac{1}{z^{n-m+1}}\frac{1}{(1-z)^{n-m+1}}\right)^{(2m)}$$

$$= \frac{(-1)^{2m+1}}{(2m)!\times 2^{2m+1}}\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q\frac{(n-m+q)!}{(n-m)!\times z^{n-m+1+q}}$$

$$\times\frac{(n-m+2m-q)!}{(n-m)!\times(1-z)^{n-m+1+2m-q}}$$

$$= \frac{(-1)^{2m+1}}{(2m)!\times 2^{2m+1}}\sum_{q=0}^{2m}\binom{2m}{q}(-1)^q\frac{(n-m+q)!}{(n-m)!\times z^{n-m+1+q}}$$

$$\times\frac{(n+m-q)!}{(n-m)!\times(1-z)^{n+m+1-q}}.$$

Evaluate this at one half to get

$$-\frac{1}{2^{2m+1}}\sum_{q=0}^{2m}\binom{n-m+q}{q}(-1)^q 2^{n-m+1+q}\binom{n+m-q}{2m-q}2^{n+m+1-q}$$

$$= -2^{2n-2m+1}\sum_{q=0}^{2m}\binom{n-m+q}{q}(-1)^q\binom{n+m-q}{2m-q}.$$

For this last sum use the integral

$$\binom{n+m-q}{2m-q} = \binom{n+m-q}{n-m} = \frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{1}{v^{2m-q+1}}\frac{1}{(1-v)^{n-m+1}}\;dv.$$

This controls the range so we can let q go to infinity in the sum to get

$$\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{1}{v^{2m+1}}\frac{1}{(1-v)^{n-m+1}}\sum_{q\ge 0}\binom{n-m+q}{q}(-1)^q v^q\;dv$$

$$= \frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{1}{v^{2m+1}}\frac{1}{(1-v)^{n-m+1}}\frac{1}{(1+v)^{n-m+1}}\;dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v^2)^{n-m+1}} \; dv = \binom{n-m+m}{m} = \binom{n}{m}.$$

We have shown that

$$2S - m \times 2 \times 2^{2n-2m} \times \binom{n}{m} = 0$$

and hence may conclude that

$$S = m \times 4^{n-m} \times \binom{n}{m}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the integral in w is zero. Recall the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

In the present case this becomes

$$\begin{aligned} & -\mathrm{Res}_{w=0} \frac{1}{w^2} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(1-2z-z/w)^2} \\ &= -\mathrm{Res}_{w=0} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(w(1-2z)-z)^2} \\ &= -\mathrm{Res}_{w=0} (1+w)^m w^m \frac{1}{(w(1-2z)-z)^2} \end{aligned}$$

which is zero by inspection.

The same procedure applied to the main integral yields

$$\begin{aligned} & -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{1}{(1-1/z)^{n-m+1}} \frac{1}{(1-2/z)^{2m+1}} \\ &= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{z^{n-m+1}}{(z-1)^{n-m+1}} \frac{z^{2m+1}}{(z-2)^{2m+1}} \\ &= -\mathrm{Res}_{z=0} z^{2n+1} \frac{1}{(z-1)^{n-m+1}} \frac{1}{(z-2)^{2m+1}} \end{aligned}$$

which is zero as well.

This was math.stackexchange.com problem 1247818.

43 Same problem, streamlined proof ($B_1 B_2 R$)

Suppose we seek to verify that

$$S = \sum_{q=0}^{n} q \binom{2n}{n+q} \binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \geq m$.

This is

$$\sum_{q=0}^{n} (n-q) \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$

which has two pieces. We use the integral

$$\binom{m+n-q-1}{2m-1} = \binom{m+n-q-1}{n-m-q} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m-q+1}} (1+w)^{m+n-q-1} \, dw.$$

Observe that this integral vanishes when $q > n-m$ and we may extend q to $2n$. We get for the first piece

$$\frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} (1+w)^{m+n-1} \sum_{q=0}^{2n} \binom{2n}{q} \frac{w^q}{(1+w)^q} \, dw$$
$$= \frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n} \, dw.$$

The second piece is the negative of

$$\sum_{q=0}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1} = \sum_{q=1}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$
$$= 2n \sum_{q=1}^{n} \binom{2n-1}{q-1} \binom{m+n-q-1}{2m-1} = 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{2m-1}$$
$$= 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{n-m-q-1}.$$

This vanishes through its integral representation when $q > n-m-1$ and we obtain

$$\frac{2n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \, dw.$$

Joining the two pieces we arrive at the single integral

$$\frac{n}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n-m+1}}\frac{1}{(1+w)^{n+1-m}}(1+2w)^{2n-1}\;dw.$$

We know the residues at zero, minus one and infinity sum to zero, where the first represents the queried sum. For the residue at minus one it is given by

$$\begin{aligned}
&\frac{n}{2\pi i}\int_{|w+1|=\gamma}\frac{1}{w^{n-m+1}}\frac{1}{(1+w)^{n+1-m}}(1+2w)^{2n-1}\;dw\\
&=\frac{n}{2\pi i}\int_{|v|=\gamma}\frac{1}{(v-1)^{n-m+1}}\frac{1}{v^{n+1-m}}(2v-1)^{2n-1}\;dv\\
&=-\frac{n}{2\pi i}\int_{|v|=\gamma}\frac{1}{(-v-1)^{n-m+1}}\frac{1}{(-v)^{n+1-m}}(-1-2v)^{2n-1}\;dv\\
&=\frac{n}{2\pi i}\int_{|v|=\gamma}\frac{1}{(1+v)^{n-m+1}}\frac{1}{v^{n+1-m}}(1+2v)^{2n-1}\;dv.
\end{aligned}$$

We see that this residue also represents the queried sum. This leaves the residue at infinity which is

$$\begin{aligned}
&\mathrm{Res}_{w=\infty}\frac{1}{w^{n-m+1}}\frac{1}{(1+w)^{n+1-m}}(1+2w)^{2n-1}\\
&=-\mathrm{Res}_{w=0}\frac{1}{w^2}w^{n-m+1}\frac{1}{(1+1/w)^{n+1-m}}(1+2/w)^{2n-1}\\
&=-\mathrm{Res}_{w=0}w^{n-m-1}\frac{w^{n+1-m}}{(1+w)^{n+1-m}}\frac{(2+w)^{2n-1}}{w^{2n-1}}\\
&=-\mathrm{Res}_{w=0}\frac{1}{w^{2m-1}}\frac{(2+w)^{2n-1}}{(1+w)^{n+1-m}}.
\end{aligned}$$

Extracting coefficients we find

$$-n\sum_{q=0}^{2m-2}\binom{2n-1}{2m-2-q}2^{2n-2m+1+q}(-1)^q\binom{n-m+q}{q}.$$

Introduce (this vanishes when $q>2m-2$)

$$\begin{aligned}
&\binom{2n-1}{2m-2-q}=\binom{2n-1}{2n+1-2m+q}\\
&=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1-q}}\frac{1}{(1-z)^{2n-2m+2+q}}\;dz
\end{aligned}$$

to get for the sum

$$
-\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{2n-2m+2}}\sum_{q\geq 0}\binom{n-m+q}{q}2^q(-1)^q\frac{z^q}{(1-z)^q}\,dz
$$

$$
\begin{aligned}
&= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{2n-2m+2}}\frac{1}{(1+2z/(1-z))^{n-m+1}}\,dz \\
&= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{n-m+1}}\frac{1}{(1+z)^{n-m+1}}\,dz \\
&= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z^2)^{n-m+1}}\,dz \\
&= -n2^{2n-2m+1}[z^{2m-2}]\frac{1}{(1-z^2)^{n-m+1}} = -n2^{2n-2m+1}[z^{m-1}]\frac{1}{(1-z)^{n-m+1}} \\
&= -n2^{2n-2m+1}\binom{n-m+m-1}{m-1}.
\end{aligned}
$$

It follows that

$$
2S - n2^{2n-2m+1}\binom{n-1}{m-1} = 0 \quad \text{or} \quad S = n4^{n-m}\frac{m}{n}\binom{n}{m}
$$

which yields

$$
S = m \times 4^{n-m} \times \binom{n}{m}
$$

as claimed.

44 Symmetry of the Euler-Frobenius coefficient (B_1EIR)

Suppose we have the coefficient of the Euler-Frobenius polynomial

$$
b_k^n = \sum_{l=1}^{k}(-1)^{k-l}l^n\binom{n+1}{k-l}
$$

and we seek to show that $b_k^n = b_{n+1-k}^n$ where $0 \leq k \leq n+1$.

First re-write this as

$$
\sum_{l=0}^{k}(-1)^l(k-l)^n\binom{n+1}{l}.
$$

Introduce the Iverson bracket

$$[[0 \le l \le k]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^l}{z^{k+1}} \frac{1}{1-z} \; dz$$

and the exponentiation integral

$$(k-l)^n = \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp((k-l)w) \; dw.$$

to get for the sum (extend the summation to $n+1$ since the Iverson bracket controls the range)

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw) \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \frac{1}{1-z} \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^l z^l \exp(-lw) \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw) \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \frac{1}{1-z} (1 - z\exp(-w))^{n+1} \; dz \; dw.$$

Evaluate this using the residues at the poles at $z = 1$ and at infinity. We obtain for $z = 1$

$$-\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(1 - \exp(-w))^{n+1} \; dw,$$

note however that $1 - \exp(-w)$ starts at w so the power starts at w^{n+1} making for a zero contribution.

We get for the residue at infinity

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{k+1} \frac{1}{1-1/z} (1 - \exp(-w)/z)^{n+1}$$

$$= -\mathrm{Res}_{z=0} z^k \frac{1}{z-1} (1 - \exp(-w)/z)^{n+1}$$

$$= \mathrm{Res}_{z=0} \frac{z^k}{z^{n+1}} \frac{1}{1-z} (z - \exp(-w))^{n+1}.$$

We need to flip the sign on this one more time since we are exploiting the fact that the residues at the three poles sum to zero. Actually extracting the coefficient we get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q} \exp(-(n+1-q)w).$$

Substitute this into the integral in w to get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(-1)^{n+1-q} \exp(-(n+1-q)w)\; dw$$

$$= -\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q}(-1)^n (n+1-k-q)^n$$

$$= \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n.$$

Using the fact that $n+1-k-q$ is zero at $q = n+1-k$ we finally obtain

$$\sum_{q=0}^{n+1-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n$$

which is precisely b^n_{n+1-k} by definition, QED.

Addendum. An alternate proof (variation on the theme from above) starts from the unmodified definition and introduces

$$\binom{n+1}{k-l} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-l+1}} (1+z)^{n+1}\; dz.$$

This controls the range so we may extend l to infinity. Introduce furthermore

$$l^n = \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(lw)\; dw.$$

These two yield for the sum

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \sum_{l\ge 0} (-1)^l z^l \exp(lw)\; dz\; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{1+z\exp(w)}\; dz\; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{z+\exp(-w)}\; dz\; dw.$$

We evaluate this using the negatives of the residues at $z = -\exp(-w)$ and at infinity. We get for $z = -\exp(-w)$

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{(-1)^k}{(-1)^{k+1}\exp(-(k+1)w)} (1-\exp(-w))^{n+1}\; dw$$

$$= -\frac{n!}{2\pi i}\int_{|w|=\epsilon} \frac{\exp(kw)}{w^{n+1}}(1-\exp(-w))^{n+1}\; dw.$$

As before the exponentiated term starts at w^{n+1} so there is no coefficient on w^n for a contribution of zero.

We get for the residue at infinity (starting from the next-to-last version of the integral)

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}(-1)^k z^{k+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{1}{1+\exp(w)/z}$$

$$= -\mathrm{Res}_{z=0}\frac{1}{z^2}(-1)^k z^{k+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{z/\exp(w)}{1+z/\exp(w)}$$

$$= -\mathrm{Res}_{z=0}(-1)^k z^{k}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{\exp(-w)}{1+z/\exp(w)}.$$

Doing the sign flip and simplifying we obtain

$$\exp(-w)(-1)^k \times \mathrm{Res}_{z=0}\frac{(1+z)^{n+1}}{z^{n-k+1}}\frac{1}{1+z/\exp(w)}.$$

Extract the residue to get

$$\exp(-w)(-1)^k \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{n-k-q}\exp(-(n-k-q)w)$$

Substitute into the integral in w to obtain

$$\sum_{q=0}^{n-k}\binom{n+1}{q}\frac{n!}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+1}}(-1)^{n-q}\exp(-(n+1-k-q)w)\; dw$$

$$= \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{n-q}(-1)^n(n+1-k-q)^n$$

$$= \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{q}(n+1-k-q)^n.$$

We have obtained b^n_{n+1-k} as before.

This was math.stackexchange.com problem 1435648.

45 A probability distribution with two parameters $(B_1 B_2)$

A sum of binomial coefficients CLXVII

Suppose we have a random variable X where

$$\mathrm{P}[X=k] = \binom{N}{2n+1}^{-1} \binom{N-k}{n}\binom{k-1}{n}$$

for $k = n+1, \ldots, N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

Sum of probabilities. This is given by

$$\binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} \binom{N-k}{n}\binom{k-1}{n}.$$

Introduce

$$\binom{N-k}{n} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n-k+1}} \frac{1}{(1-z)^{n+1}}\; dz$$

and

$$\binom{k-1}{n} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k-1}}{w^{n+1}}\; dw.$$

Observe carefully that the first integral is zero when $k > N-n$ and the second one when $1 \le k \le n$ so we may extend the range of the sum to $1 \le k$.

This gives for the sum (without the scalar)

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \sum_{k\ge 1} z^{k-1}(1+w)^{k-1}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-z(1+w)}\; dw\; dz.$$

The integral in w is

$$\frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-wz/(1-z)}\; dw$$

which yields for the integral in z

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{z^n}{(1-z)^{n+1}}\; dz$$

which is

$$\binom{N-2n-1+2n+1}{2n+1} = \binom{N}{2n+1}.$$

This confirms that the probabilities sum to one.

Expectation. This is given by

$$\mathrm{E}[X] = \binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} k \binom{N-k}{n} \binom{k-1}{n}.$$

Introduce

$$\begin{aligned} k\binom{k-1}{n} &= \frac{k!}{n! \times (k-1-n)!} = (n+1)\binom{k}{n+1} \\ &= (n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^k}{w^{n+2}}\; dw. \end{aligned}$$

The range control from this integral produces zero when $0 \le k \le n$ so we may extend the sum to zero, getting

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \sum_{k\ge 0} z^k (1+w)^k \; dw\; dz.$$

The integral in w is

$$\begin{aligned} &\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \frac{1}{1-z(1+w)}\; dw \\ &= \frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \frac{1}{1-wz/(1-z)}\; dw \end{aligned}$$

which yields for the integral in z including the factor in front

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}} \frac{z^{n+1}}{(1-z)^{n+2}}\; dz$$

which is

$$(n+1)\binom{N-2n-1+2n+2}{2n+2} = (n+1)\binom{N+1}{2n+2}.$$

We will scale this at the end, same as the variance.
Variance. Start by computing

$$\mathrm{E}[(X+1)X] = \binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} (k+1)k \binom{N-k}{n} \binom{k-1}{n}.$$

Introduce

$$(k+1)k\binom{k-1}{n} = \frac{(k+1)!}{n! \times (k-1-n)!}$$

$$= (n+2)(n+1)\binom{k+1}{n+2} = (n+2)(n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k+1}}{w^{n+3}}\; dw.$$

The range control from this integral produces zero when $0 \le k \le n$ as before so we may extend the sum to zero, getting

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}$$
$$\times \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1+w}{w^{n+3}} \sum_{k\ge 0} z^k (1+w)^k \; dw\; dz.$$

The integral in w is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1+w}{w^{n+3}}\frac{1}{1-z(1+w)}\; dw$$
$$= \frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1+w}{w^{n+3}}\frac{1}{1-wz/(1-z)}\; dw$$

which yields for the integral in z including the factor in front

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}\left(\frac{z^{n+2}}{(1-z)^{n+3}} + \frac{z^{n+1}}{(1-z)^{n+2}}\right)\; dz$$

which is

$$(n+2)(n+1)\left(\binom{N-2n-2+2n+3}{2n+3} + \binom{N-2n-1+2n+2}{2n+2}\right)$$
$$= (n+2)(n+1)\left(\binom{N+1}{2n+3} + \binom{N+1}{2n+2}\right).$$

Simplification for ease of interpretation.
We get for the expectation

$$\mathrm{E}[X] = (n+1)\frac{(N+1)!}{(N-2n-1)!(2n+2)!}\frac{(N-2n-1)!(2n+1)!}{N!}$$
$$= \frac{1}{2}(N+1).$$

We obtain furthermore

$$\mathrm{E}[(X+1)X] = (n+2)(n+1)$$
$$\times\left(\frac{(N+1)!}{(N-2n-2)!(2n+3)!} + \frac{(N+1)!}{(N-2n-1)!(2n+2)!}\right)\frac{(N-2n-1)!(2n+1)!}{N!}$$
$$= \frac{1}{2}(N+1)(n+2)\left(\frac{N-2n-1}{2n+3} + 1\right)$$

$$= \frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3}.$$

This yields for the variance

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

$$= \frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3} - \frac{1}{2}(N+1) - \frac{1}{4}(N+1)^2.$$

which simplifies to

$$\mathrm{Var}[X] = \frac{1}{4}(N+1)\frac{N-2n-1}{2n+3}.$$

This was math.stackexchange.com problem 1257644.

46 An identity involving Narayana numbers (B_1)

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

Introducing

$$G(z,u) = \sum_{p\geq 1} z^p \sum_{q\geq 0} u^q \frac{1}{p}\binom{p}{q+1}\binom{p}{q}$$

$$= \sum_{p\geq 1} \frac{1}{p} z^p \sum_{q\geq 0} u^q \binom{p}{q+1}\binom{p}{q}$$

we have by inspection that

$$A(n,k,l) = [z^n][u^l]G(z,u)^{k+1}.$$

To evaluate this introduce for the inner sum term

$$\binom{p}{q+1} = \binom{p}{p-q-1} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p-q}}(1+w)^p \; dw.$$

We get for the inner sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^p}(1+w)^p\sum_{q\geq 0}\binom{p}{q}u^q w^q\; dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^p}(1+w)^p(1+uw)^p\; dw$$

$$=\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^p}(1+w(1+u+uw)))^p\; dw.$$

Extracting the coefficient from this we get

$$[w^{p-1}]\sum_{q=0}^{p}\binom{p}{q}w^q(1+u+uw)^q$$

$$=\sum_{q=0}^{p-1}\binom{p}{q}[w^{p-1-q}](1+u+uw)^q$$

$$=\sum_{q=0}^{p-1}\binom{p}{q}\binom{q}{p-1-q}u^{p-1-q}(1+u)^{2q+1-p}.$$

This is

$$\sum_{q=0}^{p-1}\binom{p}{q+1}\binom{p-1-q}{q}u^q(1+u)^{p-1-2q}.$$

Now observe that with the factor $1/p$ from the definition,

$$\frac{1}{p}\binom{p}{q+1}\binom{p-1-q}{q}=\frac{1}{q+1}\binom{p-1}{q}\binom{p-1-q}{q}$$

$$=\frac{1}{q+1}\binom{p-1}{p-1-q}\binom{p-1-q}{q}=\frac{1}{q+1}\binom{p-1}{2q}\binom{2q}{q}.$$

where

$$C_q=\frac{1}{q+1}\binom{2q}{q}$$

is a Catalan number.

We thus get for the sum

$$\sum_{p\geq 1}z^p\sum_{q=0}^{p-1}\binom{p-1}{2q}C_q u^q(1+u)^{p-1-2q}$$

$$=z\sum_{p\geq 0}z^p\sum_{q=0}^{p}\binom{p}{2q}C_q u^q(1+u)^{p-2q}$$

$$\begin{aligned}
&= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} \sum_{p\ge q} \binom{p}{2q} z^p (1+u)^p\\
&= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} \sum_{p\ge 2q} \binom{p}{2q} z^p (1+u)^p\\
&= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} (1+u)^{2q} z^{2q} \sum_{p\ge 0} \binom{p+2q}{2q} z^p (1+u)^p\\
&= z \sum_{q\ge 0} C_q u^q z^{2q} \frac{1}{(1-z(1+u))^{2q+1}}.
\end{aligned}$$

Using the generating function of the Catalan numbers

$$Q(w) = \sum_{q\ge 0} C_q w^q = \frac{1-\sqrt{1-4w}}{2w}$$

which has functional equation

$$Q(w) = 1 + wQ(w)^2$$

we obtain

$$Q\left(\frac{uz^2}{(1-z(1+u))^2}\right) = 1 + \frac{uz^2}{(1-z(1+u))^2} Q\left(\frac{uz^2}{(1-z(1+u))^2}\right)^2$$

which is

$$G(z,u)\frac{1-z(1+u)}{z} = 1 + uG(z,u)^2.$$

Extract the coefficient in z first. We get from the functional equation

$$z = \frac{G(z,u)}{uG(z,u)^2 + (1+u)G(z,u) + 1}.$$

The coefficient extractor integral is

$$\begin{aligned}
[z^n]G(z,u)^{k+1} &= \frac{k+1}{n}[z^{n-1}]G(z,u)^k G'(z,u)\\
&= \frac{k+1}{n}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^n} G(z,u)^k G'(z,u)\; dz.
\end{aligned}$$

which becomes with $G(z,u) = v$

$$\frac{k+1}{n}\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(uv^2+(1+u)v+1)^n}{v^n} v^k \; dv.$$

Here we have used from the definition of $G(z,u)$ that $G(0,u) = 0$ so the substitution maps zero to zero. Moreover $[z^1]G(z,u) \neq 0$ so when z makes one turn so does v.

This is

$$\frac{k+1}{n}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(uv(v+1)+(v+1))^n}{v^{n-k}}\;dv.$$

Extracting the coefficient on $[u^l]$ we get

$$\frac{k+1}{n}\binom{n}{l}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{v^l(v+1)^l(v+1)^{n-l}}{v^{n-k}}\;dv$$
$$=\frac{k+1}{n}\binom{n}{l}\binom{n}{n-1-k-l}=\frac{k+1}{n}\binom{n}{l}\binom{n}{k+l+1}.$$

This is the claim, QED.

Remark. The closed form of $G(z,u)$ can be computed as follows:

$$\frac{z}{1-z(1+u)}\frac{1-\sqrt{1-4uz^2/(1-z(1+u))^2}}{2uz^2/(1-z(1+u))^2}$$
$$=\frac{z}{(1-z(1+u))^2}\frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz^2/(1-z(1+u))^2}$$
$$=\frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz}.$$

The above material incorporates data from OEIS A055151 and from OEIS A001263 on Narayana numbers. This was math.stackechange.com problem 1498014.

47 Convolution of Narayana polynomials (B_1)

This is basically a re-write of the previous entry with a more general conclusion.

Suppose we define

$$C_0^{(1)}(t)=1 \quad\text{and}\quad C_n^{(1)}(t)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1}t^k$$

and let for $m\geq 2$

$$C_n^{(m)}(t)=\sum_{q=0}^{n}C_q^{(m-1)}(t)C_{n-q}^{(1)}(t).$$

This definition is equivalent to introducing

$$G(w)=\sum_{n\geq 1}C_n^{(1)}(t)w^n$$

and letting

$$C_n^{(m)}(t) = [w^n](1+G(w))^m = [w^n]\sum_{p=0}^m \binom{m}{p} G(w)^p.$$

We seek to show that

$$C_0^{(m)}(t) = 1 \quad \text{and} \quad C_n^{(m)}(t) = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m} t^k.$$

The case n zero follows by inspection. Furthermore we have from the previous section that

$$[w^n]G(w)^{p+1} = \frac{p}{n}\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(tv(v+1)+(v+1))^n}{v^{n-p}}\; dv.$$

Summing with the binomial coefficient,

$$[w^n]\sum_{p=0}^m \binom{m}{p} G(w)^p = \frac{1}{n}\frac{1}{2\pi i}\int_{|v|=\gamma}\sum_{p=0}^m p\binom{m}{p}\frac{(tv(v+1)+(v+1))^n}{v^{n+1-p}}\; dv$$

$$= \frac{m}{n}\frac{1}{2\pi i}\int_{|v|=\gamma}\sum_{p=1}^m \binom{m-1}{p-1}\frac{(tv(v+1)+(v+1))^n}{v^{n+1-p}}\; dv$$

$$= \frac{m}{n}\frac{1}{2\pi i}\int_{|v|=\gamma}\sum_{p=0}^{m-1} \binom{m-1}{p}\frac{(tv+1)^n(1+v)^n}{v^{n-p}}\; dv$$

$$= \frac{m}{n}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(tv+1)^n(1+v)^{n+m-1}}{v^n}\; dv.$$

Extracting the coefficient on $[t^k]$,

$$\frac{m}{n}\binom{n}{k}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{v^k(1+v)^{n+m-1}}{v^n}\; dv = \frac{m}{n}\binom{n}{k}\binom{n+m-1}{n-1-k}.$$

With some binomial coefficient manipulation we get

$$\frac{m}{n}\binom{n-1}{k}\frac{n}{n-k}\binom{n+m-1}{m+k} = m\binom{n-1}{k}\frac{1}{n-k}\binom{n+m}{m+k}\frac{1}{n+m}(n-k)$$

$$= \frac{m}{n+m}\binom{n-1}{k}\binom{n+m}{k+m}.$$

This is the claim.

This was math.stackexchange.com problem 1997791.

48 A property of Legendre polynomials (B_1)

Suppose we seek to determine the constant Q in the equality

$$Q_{n,m}\left(\frac{d}{dz}\right)^{n-m}(1-z^2)^n = (1-z^2)^m\left(\frac{d}{dz}\right)^{n+m}(1-z^2)^n$$

where $n \ge m$. We will compute the coefficients on $[z^q]$ on the LHS and the RHS. Writing $1-z^2 = (1+z)(1-z)$ we get for the LHS

$$\sum_{p=0}^{n-m}\binom{n-m}{p}\binom{n}{p}p!(1+z)^{n-p}$$
$$\times\binom{n}{n-m-p}(n-m-p)!(-1)^{n-m-p}(1-z)^{m+p}$$
$$= (n-m)!(-1)^{n-m}\sum_{p=0}^{n-m}\binom{n}{p}\binom{n}{n-m-p}(1+z)^{n-p}(-1)^p(1-z)^{m+p}.$$

Extracting the coefficient we get

$$(n-m)!(-1)^{n-m}\sum_{p=0}^{n-m}\binom{n}{p}\binom{n}{n-m-p}(-1)^p$$
$$\times\sum_{k=0}^{n-p}\binom{n-p}{k}(-1)^{q-k}\binom{m+p}{q-k}.$$

We use the same procedure on the RHS and merge in the $(1-z^2)^m$ term to get

$$(n+m)!(-1)^{n+m}\sum_{p=0}^{n+m}\binom{n}{p}\binom{n}{n+m-p}(-1)^p$$
$$\times\sum_{k=0}^{n+m-p}\binom{n+m-p}{k}(-1)^{q-k}\binom{p}{q-k}.$$

Working in parallel with LHS and RHS we treat the inner sum of the LHS first, putting

$$\binom{m+p}{q-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q-k+1}}(1+z)^{m+p}\,dz$$

to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q+1}}(1+z)^{m+p}\sum_{k=0}^{n-p}\binom{n-p}{k}(-1)^{q-k}z^k\,dz$$

$$= \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{m+p} (1-z)^{n-p} \; dz.$$

Adapt and repeat to obtain for the inner sum of the RHS

$$\frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{p} (1-z)^{n+m-p} \; dz.$$

Moving on to the two outer sums we introduce

$$\binom{n}{n-m-p} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m-p+1}} (1+w)^n \; dw$$

to obtain for the LHS

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z)^n \sum_{p=0}^{n-m} \binom{n}{p} (-1)^p w^p \frac{(1+z)^p}{(1-z)^p} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z)^n \left(1 - w\frac{1+z}{1-z}\right)^n \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^m (1-z-w-wz)^n \; dz \; dw.$$

Repeat for the RHS to get

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+m+1}} (1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1-z)^m (1-z-w-wz)^n \; dz \; dw.$$

Extracting coefficients from the first integral (LHS) we write

$$(1-z-w-wz)^n = (2-(1+z)(1+w))^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+z)^k (1+w)^k 2^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{m+k}{q} (1+w)^k 2^{n-k}$$

followed by the outer one which gives

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{m+k}{q} \binom{n+k}{n-m} 2^{n-k}.$$

For the second integral (RHS) we write

$$(1-z-w-wz)^n = ((1-z)(1+w)-2w)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} (1-z)^k (1+w)^k (-1)^{n-k} 2^{n-k} w^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{q} (-1)^q (1+w)^k (-1)^{n-k} 2^{n-k} w^{n-k}$$

followed by the outer one which produces

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{q} \binom{n+k}{k+m} (-1)^{n-k} 2^{n-k}.$$

The two sums are equal up to a sign and the RHS for the coefficient on $[z^q]$ is obtained from the LHS by multiplying by

$$\frac{(n+m)!}{(n-m)!} (-1)^{n-q}.$$

Observe that powers of z that are present in the LHS and the RHS always have the same parity, the coefficients being zero otherwise (either all even powers or all odd). Therefore $(-1)^{n-q}$ is in fact a constant not dependent on q, the question is which. The leading term has degree $2n-(n-m) = n+m = (2n-(n+m))+2m$ on both sides and the sign on the LHS is $(-1)^n$ and on the RHS it is $(-1)^{n+m}$. The conclusion is that the queried factor is given by

$$Q_{n,m} = (-1)^m \frac{(n+m)!}{(n-m)!}.$$

This was math.stackexchange.com problem 2066340.

49 A sum of factorials, OGF and EGF of the Stirling numbers of the second kind (B_1)

We are given that

$$r^k (r+n)! = \sum_{m=0}^{k} \lambda_m (r+n+m)!$$

and seek to determine the λ_m independent of r. We claim and prove that

$$\lambda_m = (-1)^{k+m} \sum_{p=0}^{k-m} \binom{k}{p} {k+1-p \brace m+1} n^p.$$

With this in mind we re-write the initial condition as

$$r^k = \sum_{m=0}^{k} \lambda_m m! \binom{r+n+m}{m}.$$

We evaluate the RHS starting with λ_m using the EGF of the Stirling numbers of the second kind which in the present case says that

$${k+1-p \brace m+1} = \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

We obtain for λ_m

$$(-1)^{k+m} \sum_{p=0}^{k-m} n^p \binom{k}{p} \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

The inner term vanishes when $p \ge k+2$ but in fact even better it also vanishes when $p > k-m$ which implies $m+1 > k+1-p$ because $(\exp(z)-1)^{m+1}$ starts at $[z^{m+1}]$ and we are extracting the term on $[z^{k+1-p}]$.

Hence we may extend p to infinity without picking up any extra contributions to get

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \sum_{p \ge 0} (k+1-p) \frac{n^p z^p}{p!} \; dz.$$

This is

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} ((k+1) - nz) \exp(nz) \; dz.$$

Substitute this into the outer sum to get

$$(-1)^k \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1) - nz) \exp(nz)$$

$$\times \sum_{m=0}^{k} \binom{r+n+m}{m} (-1)^m \frac{(\exp(z)-1)^{m+1}}{m+1} \; dz.$$

We have

$$\binom{r+n+m}{m} \frac{1}{m+1} = \binom{r+n+m}{m+1} \frac{1}{r+n}$$

and hence obtain

$$\frac{(-1)^k}{r+n} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1)-nz) \exp(nz)$$

$$\times \sum_{m=0}^{k} \binom{r+n+m}{m+1} (-1)^m (\exp(z)-1)^{m+1} \; dz.$$

We may extend m to $m > k$ in the remaining sum because the term $(\exp(z)-1)^{m+1}$ as before starts at $[z^{m+1}]$ which would then be $> k+1$ but we are extracting the coefficient on $[z^{k+1}]$, which makes for a zero contribution.

Continuing we find

$$-\sum_{m\ge 0} \binom{r+n+m}{r+n-1} (-1)^{m+1} (\exp(z)-1)^{m+1}$$

$$= 1 - \frac{1}{(1-(1-\exp(z)))^{r+n}} = 1 - \exp(-(r+n)z).$$

We get two pieces on substituting this back into the main integral, the first is

$$\frac{(-1)^k}{r+n} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1)-nz) \exp(nz) \; dz$$

$$= \frac{(-1)^k}{r+n} (k+1)! \frac{n^{k+1}}{(k+1)!} - \frac{(-1)^k}{r+n} k! n \frac{n^k}{k!} = 0.$$

and the second is

$$\frac{(-1)^{k+1}}{r+n} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1)-nz) \exp(nz) \exp(-(r+n)z) \; dz$$

$$= \frac{(-1)^{k+1}}{r+n} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1)-nz) \exp(-rz) \; dz$$

$$= \frac{(-1)^{k+1}}{r+n} (k+1)! \frac{(-r)^{k+1}}{(k+1)!} - \frac{(-1)^{k+1}}{r+n} k! n \frac{(-r)^k}{k!}$$

$$= \frac{1}{r+n} (k+1)! \frac{r^{k+1}}{(k+1)!} + \frac{1}{r+n} k! n \frac{r^k}{k!}$$

$$= \frac{1}{r+n} r^{k+1} + \frac{1}{r+n} n r^k = r^k.$$

This concludes the argument.

Addendum Nov 27 2016. Markus Scheuer proposes the identity

$$\lambda_m = (-1)^{m+k} \sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k}{p} (n+1)^{k-p}.$$

To see that this is the same as what I presented we extract the coefficient on $[n^q]$ to get

$$(-1)^{m+k} \sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k}{p} \binom{k-p}{q}.$$

Now we have

$$\binom{k}{p} \binom{k-p}{q} = \frac{k!}{p!q!(k-p-q)!} = \binom{k}{q} \binom{k-q}{p}.$$

We get

$$(-1)^{m+k} \binom{k}{q} \sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k-q}{p}.$$

We now introduce

$$\binom{k-q}{p} = \binom{k-q}{k-q-p} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q-p+1}} (1+z)^{k-q} \; dz.$$

This certainly vanishes when $p > k-q$ so we may extend p to infinity, getting for the sum

$$(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q+1}} (1+z)^{k-q} \sum_{p \ge m} \left\{ {p \atop m} \right\} z^p \; dz.$$

Using the OGF of the Stirling numbers of the second kind this becomes

$$(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-q+1}} (1+z)^{k-q} \prod_{l=1}^{m} \frac{z}{1-lz} \; dz.$$

Now put $z/(1+z) = w$ to get $z = w/(1-w)$ and $dz = 1/(1-w)^2 \; dw$ to get

$$(-1)^{m+k} \binom{k}{q} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{k-q}} \frac{1-w}{w} \frac{1}{(1-w)^2} \prod_{l=1}^{m} \frac{w/(1-w)}{1-lw/(1-w)} \; dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{k-q+1}}\frac{1}{1-w}\prod_{l=1}^{m}\frac{w}{1-w-lw}\,dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{k-q+1}}\frac{1}{1-w}\prod_{l=1}^{m}\frac{w}{1-(l+1)w}\,dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{k-q+2}}\frac{w}{1-w}\prod_{l=2}^{m+1}\frac{w}{1-lw}\,dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{k-q+2}}\prod_{l=1}^{m+1}\frac{w}{1-lw}\,dw$$

$$= (-1)^{m+k}\binom{k}{q}\left\{ {k-q+1 \atop m+1} \right\}.$$

This is the claim and we are done.
This was math.stackexchange.com problem 2028293.

50 Fibonacci, Tribonacci, Tetranacci (B_1)

Suppose we seek to evaluate the following sum (with a condition on the binomial coefficient)

$$G(n,m) = \sum_{k=0}^{n}\sum_{q=0}^{k}(-1)^q\binom{k}{q}\binom{n-1-qm}{k-1}.$$

Now when $n-1-qm<0$ we usually get a non-zero value for the binomial coefficient but this is not wanted here. Therefore we have

$$G(n,m) = \sum_{k=0}^{n}\sum_{q=0}^{\lfloor (n-k)/m\rfloor}(-1)^q\binom{k}{q}\binom{n-1-qm}{k-1}.$$

If we have lost any values for q above $\lfloor (n-k)/m\rfloor$ these would render the second binomial coefficient zero. If we have added in any values for q above k the first binomial coefficient is zero there.

Now with the integral

$$\binom{n-1-qm}{k-1} = \binom{n-1-qm}{n-k-qm} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1-qm}}{z^{n-k-qm+1}}\,dz$$

we get range control because the pole vanishes when $q>(n-k)/m$ and we may extend q to infinity. We thus obtain for the inner sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n-k+1}}\sum_{q\ge 0}(-1)^q\binom{k}{q}\frac{z^{qm}}{(1+z)^{qm}}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n-k+1}}\left(1-\frac{z^m}{(1+z)^m}\right)^k\;dz$$

This yields for the outer sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n+1}}\left(1-z\left(1-\frac{z^m}{(1+z)^m}\right)\right)^{-1}$$
$$\times\left(1-z^{n+1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\right)\;dz$$

which is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+m-1}}{z^{n+1}}\left((1-z)(1+z)^m+z^{m+1}\right)^{-1}$$
$$\times\left(1-z^{n+1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\right)\;dz$$

Extracting the second component from the difference we get

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}(1+z)^{n+m-1}\left((1-z)(1+z)^m+z^{m+1}\right)^{-1}\left(1-\frac{z^m}{(1+z)^m}\right)^{n+1}\;dz$$

The pole at zero has vanished. We now have non-zero poles at $z=-1$ and from the inverted term. These depend on m and we can certainly choose ϵ small enough so that none of them are inside the contour. Therefore this term does not contribute, leaving only

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+m-1}}{z^{n+1}}\frac{1}{(1-z)(1+z)^m+z^{m+1}}\;dz.$$

The generating function $f(w)$ of these numbers is thus given by

$$f(w)=\sum_{n\ge 0}w^n\sum_{q=0}^{n}\binom{n+m-1}{n-q}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}.$$

This is

$$\sum_{q\ge 0}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}\sum_{n\ge q}w^n\binom{n+m-1}{n-q}$$

$$= \sum_{q\ge 0} w^q [z^q] \frac{1}{(1-z)(1+z)^m + z^{m+1}} \sum_{n\ge 0} w^n \binom{n+m-1+q}{n}$$

$$= \frac{1}{(1-w)^m} \sum_{q\ge 0} \frac{w^q}{(1-w)^q} [z^q] \frac{1}{(1-z)(1+z)^m + z^{m+1}}.$$

What we have here is an *annihilated coefficient extractor* that simplifies to

$$f(w) = \frac{1}{(1-w)^m} \frac{1}{(1-w/(1-w))(1+w/(1-w))^m + (w/(1-w))^{m+1}}$$

$$= \frac{1}{(1-w)^m} \frac{1}{(1-2w)/(1-w)/(1-w)^m + w^{m+1}/(1-w)^{m+1}}$$

$$= \frac{1-w}{1-2w+w^{m+1}}.$$

Now observe that

$$1 - 2w + w^{m+1} = (1-w)(1 - w - w^2 - \cdots - w^{m-1} - w^m)$$

so we finally have

$$f(w) = \left(1 - \sum_{q=1}^{m} w^q\right)^{-1} = \frac{1}{1 - w - w^2 - \cdots - w^m}.$$

We see that by the basic theory of linear recurrences what we have here is a Fibonacci, Tribonacci, Tetranacci etc. recurrence. The question is what are the initial values.

Observe however that $[w^0]f(w) = 1$ and for $1 \le q \le m$ we have

$$[w^q]\frac{1-w}{1-2w+w^{m+1}} = [w^q]\frac{1}{1-2w+w^{m+1}} - [w^{q-1}]\frac{1}{1-2w+w^{m+1}}.$$

But

$$\frac{1}{1-2w+w^{m+1}} = \frac{1}{1-2w(1-w^m/2)} = \sum_{n\ge 0} 2^n w^n (1-w^m/2)^n$$

With the condition on q and $n \ge 1$ only the constant term from the term $(1-w^m/2)^n$ contributes because the degree would be more than m otherwise. This produces just one matching term with coefficient 2^q.

This yields for $f(w)$

$$[w^q]f(w) = 2^q - 2^{q-1} = 2^{q-1}.$$

Therefore we get for the intial terms starting at $q = 0$

$$1, 1, 2, 4, 8, 16, \ldots, 2^{m-1} \quad \text{with recurrence} \quad f_n = \sum_{q=1}^{m} f_{n-q}.$$

This recurrence also shows (by subtraction) that the sequence may be produced starting from $m-1$ zero terms followed by one.

The OEIS has the Fibonacci numbers, OEIS A000045

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and the Tribonacci numbers, OEIS A000073

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, \ldots$$

and the Tetranacci numbers, OEIS A000078

$$1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \ldots$$

and more.

This was math.stackexcange.com problem 1626949.

51 Stirling numbers of two kinds, binomial coefficients

Suppose we seek to verify that

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{k} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}$$

where presumably $n \ge m$. We need for the second binomial coefficient that $m \ge q$ so this is

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}.$$

Observe that the Stirling number of the second kind vanishes when $k > n$ so we may extend the summation to infinity, getting

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k \ge m} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}.$$

Recall that

$$\left[k \atop q \right] = [w^q] k! \times \binom{w+k-1}{k}.$$

Starting with the inner sum we obtain

$$n! \sum_{q=0}^{m} \frac{1}{(m-q)!} [z^{n+q-m}] (\exp(z)-1)^k [w^q] \binom{w+k-1}{k}$$
$$= n! \sum_{q=0}^{m} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

Now when $q > m$ the coefficient extractor in w yields zero, hence we may extend the sum in q to infinity:

$$n! \sum_{q \ge 0} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

We thus obtain

$$\frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \sum_{q \ge 0} \frac{1}{q!} z^q w^q \; dw \; dz$$

$$= \frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \exp(zw) \; dw \; dz.$$

Preparing the outer sum we obtain

$$\sum_{k \ge m} \binom{k}{m} (-1)^k (\exp(z)-1)^k \binom{w+k-1}{k}$$
$$= \sum_{k \ge m} \binom{k}{m} (-1)^k (\exp(z)-1)^k [v^k] \frac{1}{(1-v)^w}.$$

Note that for a formal power series $Q(v)$ we have

$$\sum_{k \ge m} \binom{k}{m} (-1)^{k-m} u^{k-m} [v^k] Q(v) = \frac{1}{m!} \left. (Q(v))^{(m)} \right|_{v=-u}.$$

We get for the derivative in v

$$\left(\frac{1}{(1-v)^w} \right)^{(m)} = m! \binom{w+m-1}{m} \frac{1}{(1-v)^{w+m}}.$$

Substituting $u = \exp(z) - 1$ yields

$$m! \binom{w+m-1}{m} \exp(-(w+m)z).$$

Returning to the double integral we find

$$\frac{(-1)^n \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^m (-1)^m$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \exp(zw) \binom{w+m-1}{m} \exp(-(w+m)z) \; dw \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^m (-1)^m \exp(-mz)$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+m-1}{m} \; dw \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^m (-1)^m \exp(-mz) \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1-\exp(-z))^m (-1)^m \; dz$$

$$= \frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(-z)-1)^m \; dz.$$

Finally put $z=-v$ to get

$$-\frac{(-1)^n \times n!}{2\pi i \times m!} \int_{|v|=\epsilon} \frac{(-1)^{n+1}}{v^{n+1}} (\exp(v)-1)^m \; dv$$

$$= \frac{n!}{2\pi i \times m!} \int_{|v|=\epsilon} \frac{1}{v^{n+1}} (\exp(v)-1)^m \; dv.$$

This is

$$n![v^n] \frac{(\exp(v)-1)^m}{m!} = \left\{ n \atop m \right\}$$

and we have the claim.
This was math.stackexchange.com problem 1926107.

52 An identity involving two binomial coefficients and a fractional term (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{m} \frac{q}{pk+q} \binom{pk+q}{k} \binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

Observe that

$$\binom{pk+q}{k} = \frac{pk+q}{k}\binom{pk+q-1}{k-1}$$

so that

$$\binom{pk+q}{k} - p\binom{pk+q-1}{k-1} = \frac{q}{k}\binom{pk+q-1}{k-1} = \frac{q}{pk+q}\binom{pk+q}{k}.$$

This yields two pieces for the sum, call them S_1

$$\sum_{k=0}^{m}\binom{pk+q}{k}\binom{pm-pk}{m-k}$$

and S_2

$$-p\sum_{k=0}^{m}\binom{pk+q-1}{k-1}\binom{pm-pk}{m-k}.$$

For S_1 introduce the integrals

$$\binom{pk+q}{k} = \frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^{pk+q}}{z^{k+1}}\,dz$$

and

$$\binom{pm-pk}{m-k} = \frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm-pk}}{w^{m-k+1}}\,dw.$$

The second one controls the range of the sum because the pole at zero vanishes when $k > m$ so we may extend k to infinity, getting for the sum

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}\frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^{q}}{z}\sum_{k\geq 0}\frac{w^k}{z^k}\frac{(1+z)^{pk}}{(1+w)^{pk}}\,dz\,dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}\frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^{q}}{z}\frac{1}{1-w(1+z)^p/z/(1+w)^p}\,dz\,dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm+p}}{w^{m+1}}\frac{1}{2\pi i}\int_{|z|=\gamma}(1+z)^{q}\frac{1}{z(1+w)^p-w(1+z)^p}\,dz\,dw.$$

We require convergence of the geometric series i.e. $|w(1+z)^p| < |z(1+w)^p|$. This says that $\varepsilon/\gamma < |(1+w)^p/(1+z)^p|$. We also have $|(1+w)^p/(1+z)^p| > (3/5)^p$ supposing that $\varepsilon, \gamma < \frac{1}{4}$. Hence we may take $\varepsilon = 1/Q$ and $\gamma = 2^p/Q$ with $Q \geq 2^{p+3}$, Q large. Now $z = w$ is the only pole inside the contour and it is simple. The latter follows from the fact that when setting $z = w$ in the derivative

$$(1+w)^p - pw(1+z)^{p-1}\big|_{z=w}$$
$$= (1+w)^p - pw(1+w)^{p-1} = (1+w)^{p-1}(1+w-pw).$$

we can choose ε small enough such that $|1+w-wp|>0$ so the pole is order one which yields

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{(1+w)^{pm+p}}{w^{m+1}}(1+w)^q \frac{1}{(1+w)^{p-1}}\frac{1}{1+w-pw}\,dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{(1+w)^{pm+q+1}}{w^{m+1}}\frac{1}{1+w-pw}\,dw.$$

Following exactly the same procedure we obtain for S_2

$$-p\frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{(1+w)^{pm+q}}{w^{m}}\frac{1}{1+w-pw}\,dw.$$

Adding these two pieces now yields

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{(1+w)^{pm+q}}{w^{m}}\left(\frac{1+w}{w}-p\right)\frac{1}{1+w-pw}\,dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{(1+w)^{pm+q}}{w^{m+1}}\,dw = \binom{pm+q}{m}.$$

Concerning the $p-1$ other poles we can factor

$$z(1+w)^p - w(1+z)^p = \sum_{q=0}^{p}\binom{p}{q}(zw^q - wz^q) = (z-w) + zw\sum_{q=2}^{p}\binom{p}{q}(w^{q-1}-z^{q-1})$$

$$= (z-w) + zw(w-z)\sum_{q=2}^{p}\binom{p}{q}\sum_{r=0}^{q-2} z^r w^{q-2-r}.$$

Hence we need to show that

$$1 - zw\sum_{q=2}^{p}\binom{p}{q}\sum_{r=0}^{q-2} z^r w^{q-2-r}.$$

does not have any roots inside $|z|=\gamma$. Exchangeing summations and rewriting,

$$\sum_{r=0}^{p-2} z^r \sum_{q=r+2}^{p}\binom{p}{q} w^{q-2-r} = \frac{1}{zw}.$$

Choosing Q large and using that $\varepsilon, \gamma \ll 1$ we can make the RHS as large as we want in modulus yet the LHS is a finite sum bounded above by a constant in p and hence equality does not occur in this domain.

Remark Mon Jan 25 2016.

An alternate proof from the integral

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}\sum_{k\ge 0}\frac{w^k}{(1+w)^{pk}}\frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^q}{z^{k+1}}(1+z)^{pk}\;dz\;dw$$

Now put

$$u=\frac{z}{(1+z)^p}\quad\text{and introduce}\quad g(u)=z.$$

Note that the origin in z gets mapped to the origin in u.

We then have

$$du=\left(\frac{1}{(1+z)^p}-p\frac{z}{(1+z)^{p+1}}\right)\;dz=\left(\frac{u}{g(u)}-\frac{pu}{1+g(u)}\right)\;dz$$

and

$$dz=\frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\;du.$$

This yields

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}\sum_{k\ge 0}\frac{w^k}{(1+w)^{pk}}$$
$$\times\frac{1}{2\pi i}\int_{|u|=\gamma}\frac{1}{g(u)u^k}(1+g(u))^q\frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\;du\;dw$$

or

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}\;(1+g(u))^q\frac{1+g(u)}{1+g(u)-pg(u)}\bigg|_{u=w/(1+w)^p}\;dw.$$

Now observe that $g(w/(1+w)^p)=w$ by definition so we get

$$\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm}}{w^{m+1}}(1+w)^q\frac{1+w}{1+w-pw}\;dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\varepsilon}\frac{(1+w)^{pm+q+1}}{w^{m+1}}\frac{1}{1+w-pw}\;dw.$$

This is exactly the same as before and the rest of the proof continues unchanged. Concerning $g(u)$ it is analytic in a neighborhood of the origin and has a Taylor series there whose coefficients can be calculated for $n\ge 1$ (the value for $n=0$ must be set to zero):

$$\frac{1}{n}\frac{1}{2\pi i}\int_{|u|=\gamma}\frac{1}{u^n}g'(u)\;du = \frac{1}{n}\frac{1}{2\pi i}\int_{|z|=\gamma}\frac{(1+z)^{pn}}{z^n}\;dz$$
$$= \frac{1}{n}\binom{np}{n-1} = \frac{1}{n(p-1)+1}\binom{np}{n}.$$

The radius of convergence of the series can be shown to equal $(p-1)^{p-1}/p^p$ which was done at the following link which is MSE 4983957. For the ratio between consecutive terms we find

$$\frac{n(p-1)+p}{n(p-1)+1}\frac{(pn)!}{n!\times(pn-n)!}\frac{(n+1)!\times(pn+p-n-1)!}{(pn+p)!}.$$

Now we have

$$\frac{(pn)!}{(pn+p)!} = \prod_{q=1}^{p}\frac{1}{pn+q} = \frac{1}{(pn)^p}\prod_{q=1}^{p}\frac{1}{1+q/p/n}$$
$$= \frac{1}{(pn)^p}\exp\left[\sum_{q=1}^{p}\log\frac{1}{1+q/p/n}\right].$$

With n large we may replace the logarithm with the first term of its Taylor series to get

$$\frac{1}{(pn)^p}\exp\left[-\frac{1}{pn}\sum_{q=1}^{p}q\right] = \frac{1}{(pn)^p}\exp\left[-\frac{1}{2}\frac{p+1}{n}\right].$$

We also have

$$\frac{(pn+p-n-1)!}{(pn-n)!} = \prod_{q=1}^{p-1}(pn-n+q) = (pn-n)^{p-1}\prod_{q=1}^{p-1}(1+q/(p-1)/n)$$
$$= (pn-n)^{p-1}\exp\left[\sum_{q=1}^{p-1}\log(1+q/(p-1)/n)\right]$$
$$\sim (pn-n)^{p-1}\exp\left[\frac{1}{(p-1)n}\sum_{q=1}^{p-1}q\right] = (pn-n)^{p-1}\exp\left[\frac{1}{2}\frac{p}{n}\right].$$

Collecting everything,

$$\frac{(n(p-1)+p)(n+1)}{n(p-1)+1}n^{p-1}(p-1)^{p-1}\frac{1}{n^p p^p}\exp\left[-\frac{1}{2n}\right]$$
$$= \frac{n(p-1)+p}{n(p-1)+1}\frac{n+1}{n}\frac{(p-1)^{p-1}}{p^p}\exp\left[-\frac{1}{2n}\right] \sim \frac{(p-1)^{p-1}}{p^p}.$$

This will do it for the radius of convergence. Here we assumed that $g(u)$ is analytic in a neighborhood of the origin and we then computed the radius of convergence of that branch. Note also that there is a potential pole in the last integral involving $g(u)$ when $1 + g(u) - pg(u) = 0$ or $g(u) = 1/(p-1)$ but this gives $u = 1/(p-1)/(p/(p-1))^p = (p-1)^{p-1}/p^p$ and we take γ less than the radius of convergence, so this pole is not inside the contour, making for a zero contribution. Note furthermore that our choice of Q excludes the pole at $w = 1/(p-1)$ from inside the contour.

This was math.stackexchange.com problem 1620083.

53 Double chain of a total of three integrals ($B_1 B_2$)

Suppose we seek to verify that

$$\sum_{k=q}^{n-1} \frac{q}{k} \binom{2n-2k-2}{n-k-1} \binom{2k-q-1}{k-1} = \binom{2n-q-2}{n-1}.$$

This is the same as

$$\sum_{k=q}^{n} \frac{q}{k} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1} = \binom{2n-q}{n}.$$

which is equivalent to

$$\sum_{k=q}^{n} \frac{q-k}{k} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1} + \sum_{k=q}^{n} \binom{2n-2k}{n-k} \binom{2k-q-1}{k-1}$$
$$= \binom{2n-q}{n}.$$

Now

$$\frac{q-k}{k} \binom{2k-q-1}{k-1} = \frac{q-k}{k} \frac{(2k-q-1)!}{(k-1)!(k-q)!}$$
$$= -\frac{(2k-q-1)!}{k!(k-q-1)!} = -\binom{2k-q-1}{k}.$$

It follows that what we have is in fact

$$\sum_{k=q}^{n} \binom{2n-2k}{n-k} \left(\binom{2k-q-1}{k-1} - \binom{2k-q-1}{k} \right) = \binom{2n-q}{n}$$

or alternatively

$$\sum_{k=q}^{n} \binom{2n-2k}{n-k} \left(\binom{2k-q-1}{k-q} - \binom{2k-q-1}{k-q-1} \right) = \binom{2n-q}{n}.$$

There are two pieces here, call them A and B. We use the integral representation

$$\binom{2n-2k}{n-k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-2k}}{z^{n-k+1}} \; dz$$

which is zero when $k > n$ (pole vanishes) so we may extend k to infinity. We also use the integral

$$\binom{2k-q-1}{k-q} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{2k-q-1}}{w^{k-q+1}} \; dw$$

which is zero when $k < q$ so we may extend k back to zero. We obtain for piece A

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \sum_{k \ge 0} \frac{z^k}{(1+z)^{2k}} \frac{(1+w)^{2k}}{w^k} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \frac{1}{1 - z(1+w)^2/w/(1+z)^2} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{w(1+z)^2 - z(1+w)^2} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{(z-w)(z-1/w)} \; dz \; dw.$$

The derivation for piece B is the same and yields

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1}{(z-w)(z-1/w)} \; dz \; dw.$$

The difference of these two is

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{w^{q-1}}{(1+w)^{q+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+1}} \frac{1-w}{(z-w)(z-1/w)} \; dz \; dw.$$

Using partial fractions by residues we get

$$\frac{1-w}{(z-w)(z-1/w)} = \frac{1-w}{w-1/w} \frac{1}{z-w} + \frac{1-w}{1/w-w} \frac{1}{z-1/w}$$

$$= \frac{w(1-w)}{w^2-1}\frac{1}{z-w} + \frac{w(1-w)}{1-w^2}\frac{1}{z-1/w} = -\frac{w}{1+w}\frac{1}{z-w} + \frac{w}{1+w}\frac{1}{z-1/w}$$

$$= \frac{1}{1+w}\frac{1}{1-z/w} - \frac{w^2}{1+w}\frac{1}{1-wz}.$$

At this point we can see that there will be no contribution from the second term but this needs to be verified. We get for the residue in z

$$-\frac{w^2}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}w^{n-p}$$

There is no pole at zero in the outer integral for a contribution of zero. Continuing with the first term we get

$$\frac{1}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{w^{n-p}}$$

which yields

$$\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+2}}\frac{1}{w^{n-p}}\;dw$$

$$= \sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{(1+w)^{q+2}}\frac{1}{w^{n-q-p+1}}\;dw$$

$$= \sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{q+1}.$$

This is

$$\sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{n-p-q}.$$

The last integral we will be using is

$$\binom{n-p+1}{n-p-q} = \frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n-p+1}}{v^{n-p-q+1}}\;dv.$$

Observe that this is zero when $p \ge n$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n+1}}{v^{n-q+1}}\sum_{p\ge 0}\binom{2n+2}{p}(-1)^{n-q-p}\frac{v^p}{(1+v)^p}\;dv$$

$$= (-1)^{n-q}\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{(1+v)^{n+1}}{v^{n-q+1}}\left(1-\frac{v}{1+v}\right)^{2n+2}\;dv$$

$$= (-1)^{n-q} \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{1}{v^{n-q+1}} \frac{1}{(1+v)^{n+1}} \; dv$$

$$= (-1)^{n-q} (-1)^{n-q} \binom{n-q+n}{n} = \binom{2n-q}{n}.$$

This is the claim. QED.

This was math.stackexchange.com problem 1708435.

54 A summation identity with four poles (B_2)

We seek to show that

$$\sum_{m=0}^{n} (-1)^m \binom{2n+2m}{n+m} \binom{n+m}{n-m} = (-1)^n 2^{2n}.$$

The LHS is

$$[z^n](1+z)^n \sum_{m=0}^{n} (-1)^m \binom{2n+2m}{n+m} (1+z)^m z^m.$$

The coefficient extractor enforces the upper limit of the sum and we may continue with

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \sum_{m\ge 0} (-1)^m \binom{2n+2m}{n+m} (1+z)^m z^m \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{(1-w)^{n+1}}$$

$$\times \sum_{m\ge 0} (-1)^m \frac{1}{w^m} \frac{1}{(1-w)^m} (1+z)^m z^m \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{(1-w)^{n+1}} \frac{1}{1+z(1+z)/w/(1-w)} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{(1-w)^n} \frac{1}{w(1-w)+z(1+z)} \; dw \; dz$$

$$= -\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{(1-w)^n} \frac{1}{(w+z)(w-(1+z))} \; dw \; dz.$$

The contribution from the pole at $w = -z$ is

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{(-1)^n}{z^n} \frac{1}{(1+z)^n} \frac{1}{1+2z} \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n+1}} \frac{1}{1+2z} \; dz = (-1)^n [z^{2n}] \frac{1}{1+2z} = (-1)^n (-1)^{2n} 2^{2n}$$

$$= \boxed{(-1)^n 2^{2n}.}$$

This is the claim. We will document a choice of γ and ϵ so that $w = 0$ and $w = -z$ are the only poles inside the contour (pole at $w = 1$ not included, nor the pole at $w = 1 + z$.)

Now we have for the pole at $w = 0$

$$-\frac{1}{(w+z)(w-(1+z))} = \frac{1}{1+2z}\frac{1}{w+z} - \frac{1}{1+2z}\frac{1}{w-(1+z)}$$
$$= \frac{1}{z}\frac{1}{1+2z}\frac{1}{1+w/z} + \frac{1}{1+z}\frac{1}{1+2z}\frac{1}{1-w/(1+z)}.$$

We get from the first piece

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+2}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{z^{n-1-q}}\; dz$$

$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{2n+1-q}}\frac{1}{1+2z}\; dz$$

$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{2n-q-p}2^{2n-q-p}$$

$$= \sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{n-p}2^{n-p}$$

$$= (-1)^n\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}.$$

The second piece yields

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{(1+z)^{n-1-q}}\; dz$$

$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^q}{z^{n+1}}\frac{1}{1+2z}\; dz$$

$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\sum_{p=0}^{q}\binom{q}{p}(-1)^{n-p}2^{n-p}$$

$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-q}2^{n-q}\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}2^{q-p}$$

$$= -(-1)^n \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} 2^{n-q}.$$

We see that the two pieces from $w = 0$ cancel so that the contribution is zero. This almost completes the proof, we only need to choose the contour so that $w = 1$ and $w = 1 + z$ are not included. For the initial geometric series to converge we need $|1 + z|\epsilon < |1 - w|\gamma$. With ϵ and γ in a neigborhood of zero we have $|1 + z|\epsilon \le (1 + \epsilon)\epsilon$ and $(1 - \gamma)\gamma \le |1 - w|\gamma$. The series converges if $(1 + \epsilon)\epsilon < (1 - \gamma)\gamma$. Therefore a good choice is $\epsilon = 1/10$ and $\gamma = 1/5$. The contour in γ clearly includes $w = 0$ and $w = -z$ and definitely does not include $w = 1$ and $w = 1 + z$ with leftmost value $9/10$. This concludes the proof.

We are not required to simplify the sum that appears in $w = 0$, but we may do so. We get

$$S_n = \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} 2^{n-q} = 2^n [z^{n-1}] \frac{1}{1-z} \frac{1}{(1-z/2)^n}$$
$$= (-1)^{n+1} 2^{2n} \mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{z-1} \frac{1}{(z-2)^n}.$$

Residues sum to zero and the residue at infinity is zero by inspection. The residue at $z = 1$ contributes -2^{2n}. The residue at $z = 2$ requires

$$\frac{1}{(2+(z-2))^n} \frac{1}{1+(z-2)} = \frac{1}{2^n} \frac{1}{(1+(z-2)/2)^n} \frac{1}{1+(z-2)}.$$

and we get the contribution

$$(-1)^{n+1} 2^n \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} (-1)^q 2^{-q} (-1)^{n-1-q} = S_n.$$

This shows that $2S_n - 2^{2n} = 0$ or $S_n = 2^{2n-1}$.

This was math.stackexchange.com problem 3729998.

55 A summation identity over odd indices with a branch cut (B_2)

In trying to evaluate

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k}$$

we require

$$\sum_{k=0}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k} \quad \text{and} \quad \sum_{k=0}^{m} \binom{2n}{2n-k} (-1)^k \binom{2m-2n}{m-k}.$$

For the first one we find

$$\sum_{k=0}^{m} \binom{2n}{k} \binom{2m-2n}{m-k} = [z^m](1+z)^{2m-2n} \sum_{k=0}^{m} \binom{2n}{k} z^k.$$

Here the coefficient extractor enforces the range and we get

$$[z^m](1+z)^{2m-2n} \sum_{k\ge 0} \binom{2n}{k} z^k = [z^m](1+z)^{2m-2n}(1+z)^{2n}$$

$$= [z^m](1+z)^{2m} = \binom{2m}{m}.$$

This also follows from Chu-Vandermonde.
Continuing with the second piece we obtain

$$\sum_{k=0}^{m} \binom{2n}{k} (-1)^k \binom{2m-2n}{m-k} = (-1)^m \sum_{k=0}^{m} \binom{2n}{m-k} (-1)^k \binom{2m-2n}{k}$$

$$= (-1)^m \sum_{k=0}^{m} (-1)^k \binom{2m-2n}{k} [z^{2n+k-m}] \frac{1}{(1-z)^{m-k+1}}.$$

Now when $k > m$ we have $[z^{2n+k-m}](1-z)^{k-m-1} = 0$ so the coefficient extractor again enforces the range and we find

$$\frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-m+1}} \frac{1}{(1-z)^{m+1}} \sum_{k\ge 0} (-1)^k \binom{2m-2n}{k} \frac{(1-z)^k}{z^k} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-m+1}} \frac{1}{(1-z)^{m+1}} \left(1 - \frac{1-z}{z}\right)^{2m-2n} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} \frac{(1-2z)^{2m-2n}}{(1-z)^{m+1}} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} \frac{(1-2z)^{2(m+1)}}{(1-z)^{m+1}} \frac{1}{(1-2z)^{2n+2}} \; dz.$$

Now put $z(1-z)/(1-2z)^2 = w$ so that

$$z = \frac{1}{2} \pm \frac{1}{2} \frac{1}{\sqrt{1+4w}}.$$

We have that $w = z + 3z^2 + 8z^3 + \cdots$ so $z = 0$ should be mapped to $w = 0$ and in fact we work with

$$z = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1+4w}}.$$

We also see from the series expansion that the small circle around the origin $|z| = \epsilon$ is mapped to a contour that encircles $w = 0$ once and may in turn be deformed to a small circle $|w| = \gamma$. We choose the branch cut on $(-\infty, -1/4]$ so that we get analyticity in a neighborhood of the origin. We also have

$$dz = \frac{1}{(1+4w)^{3/2}}\, dw.$$

At last making the substitution we obtain

$$\frac{(-1)^m}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{1}{(1/\sqrt{1+4w})^{2n+2}} \frac{1}{(1+4w)^{3/2}}\, dw$$
$$= \frac{(-1)^m}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}}(1+4w)^{n-1/2}\, dw = (-1)^m 4^m \binom{n-1/2}{m}.$$

Collecting the two pieces we find

$$\boxed{\frac{1}{2}\binom{2m}{m} + (-1)^{m+1} 2^{2m-1}\binom{n-1/2}{m}.}$$

This was math.stackexchange.com problem 3782050.

56 A stirling number identity

We seek to evaluate (note that this is zero by inspection when $k > n + m$):

$$\sum_{j=0}^{n} (-1)^{n+j} \left[{n \atop j}\right] \left\{{m+j \atop k}\right\}$$

where $k \le n$. It is claimed that it is zero for $k < n$ and n^m for $k = n$. Using standard EGFs this becomes

$$n![z^n]\sum_{j=0}^{n}(-1)^{n+j}\frac{1}{j!}\left(\log\frac{1}{1-z}\right)^j (m+j)![w^{m+j}]\frac{(\exp(w)-1)^k}{k!}$$

$$= (-1)^n n! m! [z^n] \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log\frac{1}{1-z}\right)^j$$

$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+j+1}} \frac{(\exp(w)-1)^k}{k!}\, dw$$

$$= (-1)^n n! m! [z^n] \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z}\right)^j \frac{1}{w^j} \; dw.$$

Now $\left(\log \frac{1}{1-z}\right)^j = z^j + \cdots$ so the coefficient extractor $[z^n]$ enforces the upper limit of the sum:

$$(-1)^n n! m! [z^n] \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j\ge 0} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z}\right)^j \frac{1}{w^j} \; dw$$

$$= (-1)^n n! m! \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j\ge 0} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z}\right)^j \frac{1}{w^j} \; dw \; dz$$

$$= (-1)^n n! m! \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!} \frac{1}{(1+\frac{1}{w}\log\frac{1}{1-z})^{m+1}} \; dw \; dz$$

$$= (-1)^n n! m! \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(\exp(w)-1)^k}{k!} \frac{1}{(w+\log\frac{1}{1-z})^{m+1}} \; dw \; dz.$$

Now observe that for the geometric series in j to converge we must have $|\log\frac{1}{1-z}| < |w|$. Note that with $\log\frac{1}{1-z} = z + \cdots$ the image of $|z| = \epsilon$ makes one turn around the origin, a circle of radius ϵ plus additional lower order fluctuations. We therefore choose ϵ to shrink this pseudo-circle to be entirely contained in $|w| = \gamma$. With this choice the pole at $-\log\frac{1}{1-z}$ is inside the contour in w. We thus require

$$\frac{1}{k! \times m!} \left(\sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} \exp(qw) \right)^{(m)} = \frac{1}{k! \times m!} \sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} q^m \exp(qw).$$

Evaluating the integral in w we find

$$(-1)^n \frac{n!}{k!} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \sum_{q=0}^k \binom{k}{q} (-1)^{k-q} q^m (1-z)^q \; dz$$

which is

$$\boxed{\frac{n!}{k!} \sum_{q=0}^k \binom{k}{q} \binom{q}{n} (-1)^{k-q} q^m.}$$

Now when $k < n$ we have $\binom{q}{n} = 0$ so the entire sum vanishes as claimed. We get just one term when $k = n$ namely

$$\frac{n!}{n!} \binom{n}{n} \binom{n}{n} (-1)^{n-n} n^m = n^m$$

also as claimed. This concludes the argument.

This was math.stackexchange.com problem 3852633.

57 A Catalan-Central Binomial Coefficient Convolution

We seek to show that with

$$Q(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z} \right)^n$$

we have

$$[z^k] Q(z) = \binom{n+2k}{k}.$$

Now with the branch cut on $[1/4, \infty)$ for $\sqrt{1-4z}$ we have analyticity of $Q(z)$ in a neighborhood of the origin (note that the exponentiated term does not in fact have a pole at $z = 0$) and the Cauchy Coefficient Formula applies. We obtain

$$[z^k] Q(z) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{k+1}} \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z} \right)^n \; dz.$$

We put $\sqrt{1-4z} = w$ so that $\frac{1}{\sqrt{1-4z}} \; dz = -\frac{1}{2} \; dw$ and $z = (1-w^2)/4$. With $w = 1 - 2z - \cdots$ we get as the image of $|z| = \varepsilon$ a contour that winds around $w = 1$ counterclockwise once and may be deformed to a circle, so that we obtain

$$[z^k] Q(z) = -\frac{1}{2} \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{4^{k+1}}{(1-w^2)^{k+1}} (1-w)^n \frac{1}{2^n} \frac{4^n}{(1-w^2)^n} \; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i} \int_{|w-1|=\gamma} \frac{(w-1)^n}{(w^2-1)^{n+k+1}} \; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}} \frac{1}{(w+1)^{n+k+1}} \; dw$$

$$= \frac{(-1)^k \times 2^k}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}} \frac{1}{(1+(w-1)/2)^{n+k+1}} \; dw$$

Apply the Cauchy Residue Theorem to get

$$(-1)^k \times 2^k \times (-1)^k \frac{1}{2^k} \binom{n+2k}{n+k} = \boxed{\binom{n+2k}{k}}$$

as claimed.
This was math.stackexchange.com problem 4025969.

58 Post Scriptum additions

58.1 A trigonometric sum

Suppose we seek to evaluate

$$S = \sum_{k=1}^{m-1} \sin^{2q}(k\pi/m) = \sum_{k=0}^{m-1} \sin^{2q}(2\pi k/2/m).$$

Introducing $\zeta_k = \exp(2\pi i k/2/m)$ (root of unity) we get

$$S = \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

We also have

$$\sum_{k=m}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k \exp(2\pi i m/2/m) - 1/\zeta_k / \exp(2\pi i m/2/m))^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (-\zeta_k + 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q} = S.$$

We conclude that

$$S = \frac{1}{2} \sum_{k=0}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

Introducing

$$\begin{aligned} f(z) &= \frac{(-1)^q}{2^{2q+1}} \left(z - \frac{1}{z}\right)^{2q} \frac{2mz^{2m-1}}{z^{2m}-1} \\ &= \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q}} \frac{2mz^{2m-1}}{z^{2m}-1} \end{aligned}$$

we then have

$$S = \sum_{k=0}^{2m-1} \mathrm{Res}_{z=\zeta_k} f(z).$$

Observe that the term $(z^2-1)^{2q}$ cancels the poles at ± 1 produced by $z^{2m}-1$ which however is perfectly acceptable as they correspond to $\zeta_0 = 1$ and $\zeta_m = -1$ where $\zeta_k - 1/\zeta_k$ is zero as well.

Residues sum to zero so we obtain

$$S + \mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

Now for the residue at zero we see that when $2q - 1 < 2m - 1$ or $q < m$ the residue is zero. Otherwise we get

$$\begin{aligned} &\frac{(-1)^q}{2^{2q+1}} [z^{2q-2m}] (z^2-1)^{2q} \frac{2m}{z^{2m}-1} \\ &= \frac{(-1)^q}{2^{2q+1}} [z^{2q}] (z^2-1)^{2q} \frac{2mz^{2m}}{z^{2m}-1} \\ &= -2m \frac{(-1)^q}{2^{2q+1}} \sum_{p=0}^{q} \binom{2q}{p} (-1)^{2q-p} [z^{2q-2p}] \frac{z^{2m}}{1-z^{2m}}. \end{aligned}$$

We must have $p = q - lm$ where $l \geq 1$. This yields

$$-2m \frac{1}{2^{2q+1}} \sum_{l=1}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

This is correct even when $q < m$.

Continuing with the residue at infinity we find

$$\begin{aligned} \mathrm{Res}_{z=\infty} f(z) &= -\mathrm{Res}_{z=0} \frac{1}{z^2} f(1/z) \\ &= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1/z^2-1)^{2q}}{1/z^{2q}} \frac{2m/z^{2m-1}}{1/z^{2m}-1} \end{aligned}$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1-z^2)^{2q}}{z^{2q}} \frac{2mz}{1-z^{2m}}$$

$$= -\mathrm{Res}_{z=0} \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q+1}} \frac{2m}{1-z^{2m}}.$$

This is the same as the first residue at zero except now l starts at $l = 0$ and we obtain

$$-2m \frac{1}{2^{2q+1}} \sum_{l=0}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

Joining the two pieces we finally have

$$m \frac{1}{2^{2q}} \binom{2q}{q} + m \frac{1}{2^{2q-1}} \sum_{l=1}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

This was math.stackexchange.com problem 2051454.

58.2 A class of polynomials similar to Fibonacci and Lucas Polynomials (B_1)

Suppose we seek to collect information concerning

$$\sum_{j=-\lfloor n/p \rfloor}^{\lfloor n/p \rfloor} (-1)^j \binom{2n}{n-pj}.$$

We will construct a generating function in n with $p \geq 1$ fixed. We introduce

$$\binom{2n}{n-pj} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-pj+1}} (1+z)^{2n} \; dz.$$

Now as we examine this integral we see immediately that it vanishes if $j > \lfloor n/p \rfloor$ (pole at zero disappears). Moreover when $j < -\lfloor n/p \rfloor$ we have that $[z^{n-pj}](1+z)^{2n} = 0$ so this vanishes as well. Hence with this integral in place we may let j range from $-n$ to infinity and get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n} \sum_{j=-n}^{\infty} (-1)^j z^{pj} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n} \sum_{j=0}^{\infty} (-1)^{j-n} z^{pj-pn} \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{(p+1)n+1}} (1+z)^{2n} \frac{1}{1+z^p} \; dz.$$

We get zero for the residue at infinity, as can be seen from

$$\begin{aligned}
&\mathrm{Res}_{z=\infty} \frac{1}{z^{(p+1)n+1}} (1+z)^{2n} \frac{1}{1+z^p} \\
&= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{(p+1)n+1} \frac{(1+z)^{2n}}{z^{2n}} \frac{z^p}{1+z^p} \\
&= -\mathrm{Res}_{z=0} z^{(p-1)(n+1)} (1+z)^{2n} \frac{1}{1+z^p} = 0.
\end{aligned}$$

With residues adding to zero and introducing $\rho_k = \exp(\pi i/p + 2\pi i k/p)$ we thus obtain

$$- \sum_{k=0}^{p-1} (-1)^n \frac{1}{\rho_k^{(p+1)n+1}} (1+\rho_k)^{2n} \frac{1}{p\rho_k^{p-1}} = \frac{1}{p} \sum_{k=0}^{p-1} (-1)^n \frac{1}{\rho_k^{pn+n}} (1+\rho_k)^{2n}$$

$$= \frac{1}{p} \sum_{k=0}^{p-1} \left(\frac{1}{\rho_k} + 2 + \rho_k \right)^n .$$

At this point we can compute a generating function using the fact that

$$\sum_{q \ge 0} \rho^q z^q = \frac{1}{1-\rho z} = -\frac{1}{\rho} \frac{1}{z - 1/\rho}$$

and we obtain as a first attempt

$$G_p(z) = \frac{1}{p} \sum_{k=0}^{p-1} \frac{1}{1 - 2(1+\cos(\pi/p + 2\pi k/p))z}.$$

Observe that this correctly represents the cancelation of the pole at $z = -1$ when p is odd, contributing zero when $n \ge 1$ and $1/p$ otherwise. Furthermore note that with $\rho_k = \exp((2k+1)\pi i/p)$ we have

$$\frac{1}{\rho_{p-1-k}} = \exp(-(2(p-1-k)+1)\pi i/p) = \exp((2(k+1-p)-1)\pi i/p)$$

$$= \exp((2(k+1)-1)\pi i/p - 2\pi i) = \exp((2k+1)\pi i/p) = \rho_k$$

so the poles come in pairs with no pole at -1 when p is odd. Therefore the set of poles generated by this sum corresponds to the first $(p-1)/2$ poles when p is odd and the first $p/2$ when p is even. Joining these two we get the degree of the denominator once the sum is computed being $\lfloor p/2 \rfloor$.

This first formula enables us to compute a few of these, like for $p = 8$ we get (no complex number algebra required, basic trigonometry only)

$$G_8(z) = \frac{1 - 6z + 10z^2 - 4z^3}{1 - 8z + 20z^2 - 16z^3 + 2z^4}.$$

Looking up the coefficients we find for the denominator OEIS A034807 and for the numerator OEIS A011973 which point us to three types of polynomials, Fibonacci polynomials, Dickson polynomials and Lucas polynomials. With these data we are able to state a *conjecture* for the closed form of the generating function, which is

$$G_p(z) = \left(\sum_{q=0}^{\lfloor p/2 \rfloor} \frac{p}{p-q} \binom{p-q}{q} (-1)^q z^q \right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q} (-1)^q z^q.$$

To verify this we must show that the poles are at

$$\left(\frac{1}{\rho_k} + 2 + \rho_k \right)^{-1} \quad \text{with residue} \quad -\frac{2}{p} \left(\frac{1}{\rho_k} + 2 + \rho_k \right)^{-1}$$

where the factor two appears because the poles have been paired.

We therefore require the generating functions of the polynomials that appear in $G_p(z)$. Call the numerator $A_p(z)$ and the denominator $B_p(z)$. We first compute the auxiliary generating function

$$\begin{aligned} Q_1(t,z) &= \sum_{p \ge 0} t^p \sum_{q=0}^{\lfloor p/2 \rfloor} \binom{p-q}{q} (-1)^q z^q = \sum_{q \ge 0} (-1)^q z^q \sum_{p \ge 2q} \binom{p-q}{q} t^p \\ &= \sum_{q \ge 0} (-1)^q z^q t^{2q} \sum_{p \ge 0} \binom{p+q}{q} t^p = \sum_{q \ge 0} (-1)^q z^q t^{2q} \frac{1}{(1-t)^{q+1}} \\ &= \frac{1}{1-t} \frac{1}{1+zt^2/(1-t)} = \frac{1}{1-t+zt^2}. \end{aligned}$$

We then have $A(t,z) = tQ_1(t,z)$. With $p/(p-q) = 1 + q/(p-q)$ we get two pieces for $B(t,z)$, the first is $Q_1(t,z)$ and the second is

$$\begin{aligned} Q_2(t,z) &= \sum_{p \ge 0} t^p \sum_{q=1}^{\lfloor p/2 \rfloor} \binom{p-1-q}{q-1} (-1)^q z^q = \sum_{q \ge 1} (-1)^q z^q \sum_{p \ge 2q} \binom{p-1-q}{q-1} t^p \\ &= \sum_{q \ge 1} (-1)^q z^q t^{2q} \sum_{p \ge 0} \binom{p+q-1}{q-1} t^p = \sum_{q \ge 1} (-1)^q z^q t^{2q} \frac{1}{(1-t)^q} \\ &= -\frac{zt^2/(1-t)}{1+zt^2/(1-t)} = -\frac{zt^2}{1-t+zt^2} \end{aligned}$$

and hence we have $B(t,z) = Q_1(t,z) + Q_2(t,z)$. This yields the closed form

$$G_p(z) = \frac{[t^p]\frac{t}{1-t+zt^2}}{[t^p]\frac{1-zt^2}{1-t+zt^2}}.$$

Now introducing (we meet a shifted generating function of the Catalan numbers)

$$\alpha(z) = \frac{1+\sqrt{1-4z}}{2} \quad \text{and} \quad \beta(z) = \frac{1-\sqrt{1-4z}}{2}$$

we have a relationship that is analogous to that between Fibonacci and Lucas polynomials, namely,

$$A_p(z) = \frac{1}{\alpha(z)-\beta(z)}(\alpha(z)^p - \beta(z)^p) \quad \text{and} \quad B_p(z) = \alpha(z)^p + \beta(z)^p.$$

We now verify that $B_p(z) = 0$ for z a value from the claimed poles. Using $1/(1/\rho_k + 2 + \rho_k) = \rho_k/(1+\rho_k)^2$ ($\rho_k = -1$ is not included here) we find

$$\alpha(z) = \frac{1+\sqrt{1-4\rho_k/(1+\rho_k)^2}}{2} = \frac{1+(1-\rho_k)/(1+\rho_k)}{2} = \frac{1}{1+\rho_k}$$

and similarly

$$\beta(z) = \frac{\rho_k}{1+\rho_k}.$$

Raising to the power p we find

$$\alpha(z)^p + \beta(z)^p = \frac{1^p + \rho_k^p}{(1+\rho_k)^p} = \frac{1-1}{(1+\rho_k)^p} = 0.$$

We have located $\lfloor p/2 \rfloor$ distinct zeros here which means given the degree of $B_p(z)$ the poles are all simple. This means we may evaluate the residue by setting $z = \rho_k/(1+\rho_k)^2$ in (differentiate the denominator)

$$\frac{1}{p}\left(\sum_{q=0}^{\lfloor p/2 \rfloor} \frac{1}{p-q}\binom{p-q}{q}(-1)^q q z^{q-1}\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

which is

$$\frac{z}{p}\left(\sum_{q=1}^{\lfloor p/2 \rfloor} \binom{p-1-q}{q-1}(-1)^q z^q\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

The numerator is $A_p(z)$ and we get

$$\frac{1+\rho_k}{1-\rho_k}\frac{2}{(1+\rho_k)^p} = \frac{2}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

The denominator is $[t^p]Q_2(t,z)$ which is

$$[t^p]\frac{-zt^2}{1-t+zt^2} = [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^p]\frac{1}{1-t+zt^2}$$

$$= [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^{p+1}]\frac{t}{1-t+zt^2} = B_p(z) - A_{p+1}(z) = -A_{p+1}(z).$$

We get

$$-\frac{1+\rho_k}{1-\rho_k}\frac{1^{p+1}-\rho_k^{p+1}}{(1+\rho_k)^{p+1}} = -\frac{(1+\rho_k)^2}{(1-\rho_k)(1+\rho_k)^{p+1}} = -\frac{1}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

Joining numerator and denominator and multiplying by z/p finally produces

$$\frac{1}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}\frac{2/(1-\rho_k)/(1+\rho_k)^{p-1}}{-1/(1-\rho_k)/(1+\rho_k)^{p-1}} = -\frac{2}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}$$

as claimed. We have proved that the formula from the Egorychev method matches the conjectured form in terms of a certain class of polynomials that are related to Fibonacci and Lucas polynomials as well as Catalan numbers.

This was math.stackexchange.com problem 2237745.

58.3 Partial row sums of Pascal's triangle (B_1)

Here we seek to prove that

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k} = \sum_{k=0}^{n}\binom{m+1}{k}.$$

This is

$$[z^n]\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{m-(2k+1)}$$

$$= [z^n](1+z)^{m-1}\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{-2k}.$$

Here $[z^n]$ enforces the range of the sum and we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\sum_{k\ge 0}\binom{2k+1}{k}z^k(1+z)^{-2k}\,dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m-1}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1+w}{w} \sum_{k\ge 0} \frac{(1+w)^{2k}}{w^k} z^k (1+z)^{-2k} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m-1}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1+w}{w} \frac{1}{1-z(1+w)^2/w/(1+z)^2} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1+w}{w(1+z)^2 - z(1+w)^2} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1+w}{(1-wz)(w-z)} \; dw\; dz.$$

There is no pole at $w = 0$ here. Note however that for the geometric series to converge we must have $|z(1+w)^2| < |w(1+z)^2|$. We can achieve this by taking $\gamma = 2\epsilon$ so that

$$|z(1+w)^2| \le \epsilon(1+2\epsilon)^2 = 4\epsilon^3 + 4\epsilon^2 + \epsilon\Big|_{\epsilon=1/20} = \frac{242}{4000}$$

and

$$|w(1+z)^2| \ge 2\epsilon(1-\epsilon)^2 = 2\epsilon^3 - 4\epsilon^2 + 2\epsilon\Big|_{\epsilon=1/20} = \frac{361}{4000}.$$

With these values the pole at $w = z$ is inside the contour and we get as the residue

$$\frac{1+z}{1-z^2} = \frac{1}{1-z}.$$

This yields on substitution into the outer integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{1-z} \; dz = [z^n]\frac{(1+z)^{m+1}}{1-z}$$

$$= \sum_{k=0}^{n} [z^k](1+z)^{m+1} [z^{n-k}]\frac{1}{1-z} = \sum_{k=0}^{n} {m+1 \choose k}.$$

This is the claim.

Remark. For the pole at $w = 1/z$ to be inside the contour we would need $1/\epsilon < 2\epsilon$ or $1 < 2\epsilon^2$ which does not hold here so this pole does not contribute.

This was math.stackexchange.com problem 3640984.

58.4 The Tree function and Eulerian numbers of the second order

We seek to show that the following identity holds:

$$2^{n+1} \sum_{k=0}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle \frac{1}{2^k} = n![x^n] \frac{1}{1+W(-\exp((x-1)/2)/2)}.$$

We will be using data from Wikipedia on Lambert W and work with the combinatorial branch which is $W_0(z)$.

Recall that

$$W'(z)\frac{z}{W(z)} = \frac{1}{1+W(z)}.$$

We obtain

$$[z^m]\frac{1}{1+W(z)} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^m}\frac{1}{W(z)}W'(z)\;dz.$$

Putting $W(z) = v$ we find

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^m \exp(mv)}\frac{1}{v}\;dv = \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^{m+1}}\exp(-mv)\;dv = \frac{(-1)^m m^m}{m!}.$$

so that

$$\frac{1}{1+W(z)} = \sum_{m\ge 0} (-1)^m m^m \frac{z^m}{m!}.$$

We get for the original RHS

$$\begin{aligned}
&n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\exp(m(x-1)/2)\frac{1}{2^m}\\
&= n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\frac{\exp(-m/2)}{2^m}\exp(mx/2)\\
&= \sum_{m\ge 0}\frac{m^{m+n}}{m!}\frac{\exp(-m/2)}{2^{m+n}}.
\end{aligned}$$

First part. Introduce the tree function $T(z)$ from combinatorics where $T(z) = z\exp T(z)$ and $T(z) = -W_0(-z)$. Note that we have by Cayley's theorem that $T(z) = \sum_{m\ge 1} m^{m-1}\frac{z^m}{m!}$. We claim that with $n \ge 1$

$$Q_n(z) = \sum_{m\ge 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k.$$

This means the RHS is $\frac{1}{2^n}Q_n(\exp(-1/2)/2)$. To verify this last identity note that $Q_{n+1}(z) = z\frac{d}{dz}Q_n(z)$ so we may prove it by induction.

We get for the RHS of the series identity on differentiating and multiplying by z

$$\frac{(2n+1)zT'(z)}{(1-T(z))^{2n+2}} \sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + \frac{z}{(1-T(z))^{2n+1}} \sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^{k-1}T'(z)$$

Extracting the term $zT'(z)/(1-T(z))^{2n+2}$ in front leaves us with

$$(2n+1)\sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + (1-T(z)) \sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^{k-1}$$

$$= (2n+1)\sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + \sum_{k=0}^{n-1} \left\langle\!\!\left\langle n \atop k+1 \right\rangle\!\!\right\rangle (k+1)T(z)^k - \sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^k$$

$$= \sum_{k=1}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle (2n+2-(k+1))T(z)^k + \sum_{k=0}^{n-1} \left\langle\!\!\left\langle n \atop k+1 \right\rangle\!\!\right\rangle (k+1)T(z)^k.$$

We may include $k=0$ in the first sum and $k=n$ in the second. Now the Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle n-1 \atop k \right\rangle\!\!\right\rangle k + \left\langle\!\!\left\langle n-1 \atop k-1 \right\rangle\!\!\right\rangle (2n-k)$$

We have shown that

$$Q_{n+1}(z) = \frac{zT'(z)}{(1-T(z))^{2n+2}} \sum_{k=0}^{n} \left\langle\!\!\left\langle n+1 \atop k+1 \right\rangle\!\!\right\rangle T(z)^k$$

$$= \frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} \sum_{k=1}^{n+1} \left\langle\!\!\left\langle n+1 \atop k \right\rangle\!\!\right\rangle T(z)^k.$$

Now we just have to verify that

$$\frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} = \frac{1}{(1-T(z))^{2n+3}} \quad \text{or} \quad zT'(z)(1-T(z)) = T(z).$$

The functional equation tells us that $T'(z) = \exp T(z) + z\exp T(z)T'(z)$ so that $T'(z)(1-T(z)) = \exp T(z) = T(z)/z$ which is just what we need. It remains to verify the base case so the induction starts properly. We seek

$$Q_1(z) = \sum_{m\ge 0} m^{m+1}\frac{z^m}{m!} = \frac{T(z)}{(1-T(z))^3}.$$

We verify this by coefficient extraction. We get

$$m![z^m]Q_1(z) = \frac{m!}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{T(z)}{(1-T(z))^3}\; dz.$$

With $T(z) = z + \cdots$ this integral will produce the correct value zero for $m = 0$. For $m \ge 1$, we put $T(z) = w$ so that $z = w\exp(-w)$ and $dz = \exp(-w)(1-w)\; dw$ and obtain

$$\frac{m!}{2\pi i}\int_{|w|=\gamma} \frac{\exp((m+1)w)}{w^{m+1}} \frac{w}{(1-w)^3} \exp(-w)(1-w)\; dw$$
$$= \frac{m!}{2\pi i}\int_{|w|=\gamma} \frac{\exp(mw)}{w^{m}} \frac{1}{(1-w)^2}\; dw.$$

This is

$$m! \sum_{q=0}^{m-1} \frac{m^q}{q!}(m-q) = m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=1}^{m-1} \frac{m^q}{(q-1)!}$$
$$= m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=0}^{m-2} \frac{m^{q+1}}{q!} = m! \frac{m^m}{(m-1)!} = m^{m+1}$$

as desired.

Sequel. Note that in the identity for $Q_n(z)$ we have by the definition of the Eulerian numbers that $\left\langle\!\!\left\langle {n \atop 0} \right\rangle\!\!\right\rangle$ is zero when $n \ge 1$. Therefore we may extend k to include zero (with $n \ge 1$ for the moment) which yields

$$\boxed{Q_n(z) = \sum_{m\ge 0} m^{m+n} \frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k.}$$

Now observe that this will produce $Q_0(z) = \sum_{m\ge 0} m^m \frac{z^m}{m!} = \frac{1}{1-T(z)}$ due to $\left\langle\!\!\left\langle {0 \atop 0} \right\rangle\!\!\right\rangle = 1$ which is in fact correct because unlike $Q_n(z)$ with $n \ge 1$, $Q_0(z)$ has a constant term, which is one (this is because $m^{m+n} = 0$ for $m = 0$ and $n \ge 1$ and $m^{m+n} = 1$ for $m = 0$ and $n = 0$). Therefore

$$Q_0(z) = 1 + zT'(z) = 1 + \frac{T(z)}{1-T(z)} = \frac{1}{1-T(z)}$$

as obtained from the boxed version of the main identity, which is seen to hold for all $n \ge 0$.

Conclusion. We are now ready to answer the original question. We have shown that the RHS is $\frac{1}{2^n} Q_n(\exp(-1/2)/2)$. By our formula for $Q_n(z)$ in terms of the tree function we obtain with $T(\exp(-1/2)/2) = \frac{1}{2}$ at last the closed form

$$\frac{1}{2^n} \frac{1}{(1-1/2)^{2n+1}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k} = 2^{n+1} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k}$$

which is the LHS and hence the claim.

This was math.stackexchange.com problem 4040942.

58.5 A Stirling set number generating function and Eulerian numbers of the second order

Supposing that $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind giving the count of partitions of a set of n distinguishable objects into k non-empty subsets we seek to show that

$$\left\{ {n+r \atop n} \right\}$$

is a polynomial of degree $2r$ in n. We start with the following claim for $r \ge 0$:

$$\boxed{Q_r(z) = \sum_{n\ge 0} \left\{ {n+r \atop n} \right\} z^n = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle z^k.}$$

We will prove this by induction. Note that depending on whether ball $n+r+1$ joins an existing set or becomes a singleton we have

$$\left\{ {n+r+1 \atop n} \right\} = n \left\{ {n+r \atop n} \right\} + \left\{ {n+r \atop n-1} \right\}.$$

Multiply by z^n and sum over $n \ge 0$ to get

$$Q_{r+1}(z) = zQ_r'(z) + \sum_{n\ge 1} \left\{ {n+r \atop n-1} \right\} z^n = zQ_r'(z) + z \sum_{n\ge 0} \left\{ {n+r+1 \atop n} \right\} z^n$$

$$= zQ_r'(z) + zQ_{r+1}(z).$$

This means we have

$$Q_{r+1}(z) = \frac{z}{1-z} Q_r'(z).$$

Now to prove the claim it certainly holds for $r = 0$ by inspection. It also holds for $r = 1$ since

$$\sum_{n\ge 1} \binom{n+1}{2} z^n = z \sum_{n\ge 1} \binom{n+1}{2} z^{n-1} = z \sum_{n\ge 0} \binom{n+2}{2} z^n = \frac{z}{(1-z)^3}.$$

For the induction step supposing it holds for $r \ge 1$ we differentiate and multiply by $z/(1-z)$ to get for $Q_{r+1}(z)$

$$\frac{z}{1-z} \frac{2r+1}{(1-z)^{2r+2}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle z^k + \frac{z}{1-z} \frac{1}{(1-z)^{2r+1}} \sum_{k=1}^{r} \left\langle\!\!\left\langle {r \atop k} \right\rangle\!\!\right\rangle k z^{k-1}.$$

Factoring out $1/(1-z)^{2r+3}$ for the moment we are left with

$$(2r+1)\sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle z^{k+1} + (1-z)\sum_{k=1}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k$$

$$= (2r+1)\sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle z^{k} + \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k - \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^{k+1}$$

$$= (2r+1)\sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle z^{k} + \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle kz^k - \sum_{k=1}^{r+1} \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle (k-1)z^{k}$$

Now with $r \geq 1$ we may extend the first and the third sum to include $k = 0$ and the second to include $k = r + 1$ to obtain

$$\sum_{k=0}^{r+1} \left[(2r+2-k) \left\langle\!\!\left\langle r \atop k-1 \right\rangle\!\!\right\rangle + k \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle \right] z^k.$$

The Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = k \left\langle\!\!\left\langle n-1 \atop k \right\rangle\!\!\right\rangle + (2n-k) \left\langle\!\!\left\langle n-1 \atop k-1 \right\rangle\!\!\right\rangle$$

so this is with the factor in front

$$\frac{1}{(1-z)^{2r+3}} \sum_{k=0}^{r+1} \left\langle\!\!\left\langle r+1 \atop k \right\rangle\!\!\right\rangle z^k$$

and the induction goes through.

Now to see that ${n+r \brace n}$ is a polynomial in n of degree $2r$ we extract the coefficient on $[z^n]$ of $Q_r(z)$ to get

$$\sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle \binom{2r+n-k}{2r} = \frac{1}{(2r)!} \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle (n+2r-k)^{\underline{2r}}.$$

The sum terms are products of $2r$ linear terms in n times a coefficient that does not depend on n (Eulerian number) and neither does the range of the sum (finite, $r + 1$ terms) and we have the claim. The coefficient on n^{2r} is

$$\frac{1}{(2r)!} \sum_{k=0}^{r} \left\langle\!\!\left\langle r \atop k \right\rangle\!\!\right\rangle = \frac{1}{(2r)!}(2r-1)!! = \frac{1}{(2r)!}\frac{(2r)!}{2^r r!} = \frac{1}{2^r r!} \neq 0.$$

This was math.stackexchange.com problem 4121168.

58.5.1 A Stirling cycle number generating function and Eulerian numbers of the second order (II)

We start with the following claim

$$\boxed{Q_r(z) = \sum_{n\ge 0} \left[{n+r+1 \atop n+1} \right] z^n = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^k.}$$

We will prove this by induction. Introduce $P_r(z) = z^r Q_r(z)$. Note that depending on whether ball $n+r+2$ joins an existing cycle or turns into a fixed point we have

$$\left[{n+r+2 \atop n+2} \right] = (n+r+1) \left[{n+r+1 \atop n+2} \right] + \left[{n+r+1 \atop n+1} \right].$$

Multiply by z^{n+r} and sum over $n \ge 0$ to get

$$\sum_{n\ge 0} \left[{n+r+2 \atop n+2} \right] z^{n+r} = \sum_{n\ge 0} (n+r+1) \left[{n+r+1 \atop n+2} \right] z^{n+r} + P_r(z).$$

The first term is

$$\frac{1}{z}(P_r(z) - r! z^r)$$

and the second one

$$\left(z \sum_{n\ge 0} \left[{n+2+(r-1) \atop n+2} \right] z^{n+1+r-1} \right)'$$
$$= (-(r-1)! z^r + z P_{r-1}(z))' = -r! z^{r-1} + P_{r-1}(z) + z P'_{r-1}(z).$$

This gives the recurrence

$$P_r(z) - r! z^r = -r! z^r + z P_{r-1}(z) + z^2 P'_{r-1}(z) + z P_r(z).$$

We obtain

$$P_r(z) = \frac{z}{1-z}(P_{r-1}(z) + z P'_{r-1}(z)) = \frac{z}{1-z}(z P_{r-1}(z))'.$$

We now prove by induction that

$$P_r(z) = \frac{1}{(1-z)^{2r+1}} \sum_{k=0}^{r} \left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+k}.$$

It certainly holds for $r = 0$ where the infinite series gives $1/(1-z)$ and it also holds at $r = 1$ as well where the sum gives

$$\sum_{n\ge 1} \binom{n+1}{2} z^n = z \sum_{n\ge 0} \binom{n+2}{2} z^n = \frac{z}{(1-z)^3}$$

and the Eulerian numbers produce

$$\frac{1}{(1-z)^3}\left[\left\langle\!\!\left\langle {1 \atop 1} \right\rangle\!\!\right\rangle z + \left\langle\!\!\left\langle {1 \atop 0} \right\rangle\!\!\right\rangle z^2\right] = \frac{z}{(1-z)^3}.$$

Now supposing it holds with $r \ge 1$ we must show that it holds for $r+1$. Doing the differentiation and multiplication we obtain

$$\frac{z}{1-z}\frac{2r+1}{(1-z)^{2r+2}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+1+k}$$

$$+\frac{z}{1-z}\frac{1}{(1-z)^{2r+1}}\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+k}.$$

Factoring out $1/(1-z)^{2r+3}$ for the moment this becomes

$$z(2r+1)\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+1+k} + (z-z^2)\sum_{k=0}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+k}.$$

or

$$(2r+1)\sum_{k=-1}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle z^{r+2+k} + \sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+1+k}$$

$$-\sum_{k=-1}^{r}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+2+k}.$$

Here we have included three zero terms, one in every sum. Continuing,

$$(2r+1)\sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle z^{r+1+k} + \sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle (r+1+k)z^{r+1+k}$$

$$-\sum_{k=0}^{r+1}\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle (r+k)z^{r+1+k}.$$

We obtain

$$\sum_{k=0}^{r+1}\left[(r+1-k)\left\langle\!\!\left\langle {r \atop r+1-k} \right\rangle\!\!\right\rangle + (r+1+k)\left\langle\!\!\left\langle {r \atop r-k} \right\rangle\!\!\right\rangle\right] z^{r+1+k}.$$

The Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = k \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-k) \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle$$

Putting $n := r+1$ and $k := r+1-k$ and restoring the factor in front now yields

$$\frac{1}{(1-z)^{2r+3}} \sum_{k=0}^{r+1} \left\langle\!\!\left\langle {r+1 \atop r+1-k} \right\rangle\!\!\right\rangle z^{r+1+k}$$

thus concluding the induction.

Addendum. The reader might well wonder how the conjecture from the beginning was obtained i.e. how we find the closed form for small r for lookup in the OEIS, which then points us to Eulerian numbers, enabling the whole computation.

Recall e.g. from *Concrete Mathematics* chapter 6.2. [GKP89] that

$$\boxed{\left[{n \atop m} \right] = \frac{(n-1)!}{(m-1)!} [w^{n-m}] \left(\frac{w \exp(w)}{\exp(w)-1} \right)^n .}$$

We get for our series

$$Q_r(z) = [w^r] \sum_{n \ge 0} z^n \frac{(n+r)!}{n!} \left(\frac{w}{1-\exp(-w)} \right)^{n+r+1}$$

$$= r![w^r] \left(\frac{w}{1-\exp(-w)} \right)^{r+1} \sum_{n \ge 0} z^n \binom{n+r}{r} \left(\frac{w}{1-\exp(-w)} \right)^n$$

$$= r![w^r] \left(\frac{w}{1-\exp(-w)} \right)^{r+1} \frac{1}{(1-zw/(1-\exp(-w)))^{r+1}}$$

$$= r![w^r] \frac{w^{r+1}}{(1-\exp(-w)-zw)^{r+1}}.$$

Note that the fraction is a formal power series in w with no pole at zero. Continuing,

$$r! \operatorname*{res}_w \frac{1}{(1-\exp(-w)-zw)^{r+1}}.$$

A CAS like Maple for example can recognize the pole of order $r+1$ at zero which has now appeared and quickly compute the residue by differentiation. This will produce e.g.

$$Q_5(z) = \frac{z^4 + 52\, z^3 + 328\, z^2 + 444\, z + 120}{(1-z)^{11}}$$

which is enough to spot the pattern.

This was math.stackexchange.com problem 4480877.

58.6 Another case of factorization

In seeking to evaluate

$$\sum_{\ell=0}^{\lfloor k/2\rfloor} \binom{q/2+\ell}{2\ell} \left(\binom{q/2-j+k-\ell}{k-2\ell} + \binom{q/2-j+k-\ell-1}{k-2\ell} \right)$$

We get for the first piece of the sum

$$\sum_{\ell=0}^{\lfloor k/2\rfloor} \binom{q/2+\ell}{2\ell} \binom{q/2-j+k-\ell}{k-2\ell}$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{k+1}} (1+z)^{q/2-j+k} \sum_{\ell=0}^{\lfloor k/2\rfloor} \binom{q/2+\ell}{2\ell} \frac{z^{2\ell}}{(1+z)^\ell} \; dz.$$

Now here the residue vanishes when $2\ell > k$ so it enforces the upper limit of the sum and we obtain

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{q/2}}{w} \sum_{\ell\ge 0} \frac{z^{2\ell}}{(1+z)^\ell} \frac{(1+w)^\ell}{w^{2\ell}} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{q/2}}{w} \frac{1}{1-z^2(1+w)/(1+z)/w^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{q/2-j+k+1}}{z^{k+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} (1+w)^{q/2} \frac{w}{(w-z)(w(1+z)+z)} \; dw \; dz.$$

The pole at $w=0$ has been canceled. Now observe that for the geometric series to converge we must have

$$|z^2(1+w)/w^2/(1+z)| < 1.$$

We will choose a contour that includes both simple poles. The first pole is at $-z/(1+z)$. We thus require $|z/(1+z)| < \gamma$. With $|z/(1+z)| \le \varepsilon/(1-\varepsilon)$ we get $\varepsilon/(1-\varepsilon) < \gamma$ and we furthermore need $|z^2/(1+z)| < |w^2/(1+w)|$. The latter holds if $\varepsilon^2/(1-\varepsilon) < \gamma^2/(1+\gamma)$. Both hold if $\varepsilon\gamma < \gamma^2/(1+\gamma)$ or $\varepsilon < \gamma/(1+\gamma)$. So $\varepsilon = \gamma^2/(1+\gamma)$ will work. Observe that this contour also includes the pole at $w=z$.

First pole. Now to extract the residue at $w = -z/(1+z)$ we write

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q/2-j+k}}{z^{k+1}}$$
$$\times\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{q/2}\frac{w}{(w-z)(w+z/(1+z))}\;dw\;dz$$

and obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q/2-j+k}}{z^{k+1}}(1+z)^{-q/2}\frac{-z/(1+z)}{-z/(1+z)-z}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{k-j}}{z^{k+1}}\frac{1}{z+2}\;dz.$$

Repeating for the second sum we get

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{k-j-1}}{z^{k+1}}\frac{1}{z+2}\;dz.$$

Adding the two we find

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{k-j-1}(1+(1+z))}{z^{k+1}}\frac{1}{z+2}\;dz={k-j-1 \choose k}.$$

Second pole. For the residue at $w=z$ we obtain for the first sum

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q/2-j+k+1}}{z^{k+1}}(1+z)^{q/2}\frac{z}{(z(1+z)+z)}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q-j+k+1}}{z^{k+1}}\frac{1}{z+2}\;dz.$$

Repeating for the second sum we get

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q-j+k}}{z^{k+1}}\frac{1}{z+2}\;dz.$$

Adding the two we find

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{q-j+k}(1+(1+z))}{z^{k+1}}\frac{1}{z+2}\;dz={q-j+k \choose k}.$$

Conclusion. Collecting everything we obtain

$${q-j+k \choose k}+{k-j-1 \choose k}.$$

The second term is $(k-j-1)^{\underline{k}}/k!$. Now if $0\le j<k$ this is indeed zero because the falling factorial hits the zero value. If $j\ge k$ all k terms are negative and we get $(-j)^{\overline{k}}/k!$.

We have at last

$$\boxed{\binom{q-j+k}{k} + (-1)^k \binom{j}{k}.}$$

as claimed.

Remark. The potential square roots that appeared in the above all use the principal branch of the logarithm with branch cut $(-\infty, -1]$ which means everything is analytic in a neighborhood of zero as required.

This was math.stackexchange.com problem 4155443.

58.7 An additional case of factorization

Supposing we seek to simplify

$$\sum_{j=0}^{k} \binom{2j}{j+q}\binom{2k-2j}{k-j}.$$

where $0 \le q \le k$. This is

$$[z^k](1+z)^{2k} \sum_{j=0}^{k} \binom{2j}{j+q} \frac{z^j}{(1+z)^{2j}}.$$

Here the coefficient extractor enforces the upper limit of the sum and we find

$$[z^k](1+z)^{2k} \sum_{j\ge 0} \binom{2j}{j+q} \frac{z^j}{(1+z)^{2j}}.$$

At this point we see that we will require residues and complex integration and continue with

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}} \sum_{j\ge 0} \frac{(1+w)^{2j}}{w^j} \frac{z^j}{(1+z)^{2j}} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q+1}} \frac{1}{1-z(1+w)^2/w/(1+z)^2} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k+2}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q}} \frac{1}{w(1+z)^2 - z(1+w)^2} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{2k+2}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{q}} \frac{1}{(w-z)(1-wz)} \; dw\; dz.$$

For the geometric series to converge we must have $|z(1+w)^2/w/(1+z)^2| < 1$ or $|z/(1+z)^2| < |w/(1+w)^2|$. This requires $\varepsilon/(1-\varepsilon)^2 < \gamma/(1+\gamma)^2$. We will also require $w = z$ to be inside the contour for w so we need $\varepsilon < \gamma$. With $\varepsilon \ll 1$ and $\gamma \ll 1$ we may take $\varepsilon = \gamma^2$ for the latter inquality. We then get for the inquality from the geometric series $\gamma^2/(1-\gamma^2)^2 < \gamma/(1+\gamma)^2$ or $\gamma < (1-\gamma^2)^2/(1+\gamma)^2$ or $\gamma < (1-\gamma)^2$. This holds for $\gamma < 1 - 1/\varphi$ with φ the golden mean.

Now we have the pole at zero and the one at $w = z$ inside the contour in w. This means we can evaluate the integral by using the fact that residues sum to zero, taking minus the residue at $w = 1/z$ and minus the residue at infinity, which is zero by inspection, however. (The pole at $w = 1/z$ has modulus $1/\varepsilon$ and is outside the contour.) Computing minus the residue at $w = 1/z$ we write

$$-\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k+2}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^q}\frac{1}{(w-z)(w-1/z)}\;dw\;dz.$$

With the sign change we obtain

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k+2}}z^q\frac{1}{1/z-z}\;dz = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+2}}{z^{k-q+1}}\frac{1}{1-z^2}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2k+1}}{z^{k-q+1}}\frac{1}{1-z}\;dz.$$

This is zero when $q > k$ and otherwise

$$\sum_{j=0}^{k-q}\binom{2k+1}{j} = \sum_{j=0}^{k}\binom{2k+1}{j} - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}$$

or alternatively

$$\boxed{4^k - \sum_{j=k-q+1}^{k}\binom{2k+1}{j}}$$

which is a closed form term plus a sum of q terms. E.g. with $q = 0$ we obtain 4^k and with $q = 1$, $4^k - \binom{2k+1}{k}$. For $q = 2$ we have $4^k - \binom{2k+1}{k-1} - \binom{2k+1}{k}$ and so on.

This was math.stackexchange.com problem 4174584.

58.8 Contours and a binomial square root

Suppose we seek to prove that

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n} = \binom{2m}{2n}.$$

Introduce the integral representation

$$\binom{m+k}{2n} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{2n+1}}(1+z)^{m+k}\; dz.$$

This gives the following integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \sum_{k=0}^{n} \binom{2n+1}{2k+1} \frac{1}{z^{2n+1}}(1+z)^{m+k}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^m}{z^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (1+z)^{k}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{m-1/2}}{z^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{2k+1} \sqrt{1+z}^{2k+1}\; dz.$$

The sum is

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \sqrt{1+z}^{k} \frac{1}{2}(1-(-1)^k)$$

$$= \frac{1}{2}((1+\sqrt{1+z})^{2n+1} - (1-\sqrt{1+z})^{2n+1})$$

and we get for the integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{(1+z)^{m-1/2}}{2z^{2n+1}} \left((1+\sqrt{1+z})^{2n+1} - (1-\sqrt{1+z})^{2n+1}\right)\; dz.$$

By way of ensuring analyticity we observe that we must have $\varepsilon < 1$ owing to the branch cut $(-\infty, -1]$ of the square root. Now put $1 + z = w^2$ so that $dz = 2w\; dw$ and the integral becomes

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{w^{2m-1}}{(w^2-1)^{2n+1}} \left((1+w)^{2n+1} - (1-w)^{2n+1}\right)\; w\; dw.$$

This is

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} w^{2m} \left(\frac{1}{(w-1)^{2n+1}} + \frac{1}{(w+1)^{2n+1}}\right)\; dw.$$

Treat the two terms in the parentheses in turn. The first contributes

$$[(w-1)^{2n}]w^{2m} = [(w-1)^{2n}] \sum_{q=0}^{2m} \binom{2m}{q} (w-1)^q = \binom{2m}{2n}.$$

The second term is analytic on and inside the circle that w traces round the value 1 with no poles (pole is at $w = -1$) and hence does not contribute anything. This concludes the argument.

Remark. We must document the choice of γ so that $|w-1| = \gamma$ is entirely contained in the image of $|z| = \varepsilon$, which since $w = 1 + \frac{1}{2}z + \cdots$ makes one turn around $w = 1$ and may then be continuously deformed to the circle $|w-1| = \gamma$. We need a bound on where this image comes closest to one. We have $w = 1 + \frac{1}{2}z + \sum_{q\ge 2}(-1)^{q+1}\frac{1}{4^q}\frac{1}{2q-1}\binom{2q}{q}z^q$. The modulus of the series term is bounded by $\sum_{q\ge 2}\frac{1}{4^q}\frac{1}{2q-1}\binom{2q}{q}|z|^q = 1 - \frac{1}{2}|z| - \sqrt{1-|z|}$. Therefore choosing $\gamma = \frac{1}{2}\varepsilon - 1 + \frac{1}{2}\varepsilon + \sqrt{1-\epsilon} = \sqrt{1-\varepsilon} + \varepsilon - 1$ will fit the bill. For example with $\varepsilon = 1/2$ we get $\gamma = (\sqrt{2}-1)/2$. It is a matter of arithmetic to verify that with the formula we have $\gamma < 1$.

This was math.stackexchange.com problem 601940.

58.9 Careful examination of a contour

We seek to show that

$$\sum_{q=0}^{n}\binom{q}{n-q}(-1)^{n-q}\binom{2q+1}{q+1} = 2^{n+1} - 1.$$

The LHS is

$$\frac{(-1)^n}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\sum_{q=0}^{n}(-1)^q z^q(1+z)^q\frac{(1+w)^{2q+1}}{w^{q+2}}\,dw\,dz.$$

There is no contribution when $q > n$ and we may extend q to infinity:

$$\frac{(-1)^n}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w^2}\frac{1}{1+z(1+z)(1+w)^2/w}\,dw\,dz$$

$$= \frac{(-1)^n}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\frac{1}{(1+z+wz)(z+(1+z)w)}\,dw\,dz.$$

Now we determine ε and γ so that the geometric series converges and the pole at $w = -z/(1+z)$ is inside $|w| = \gamma$ while the pole at $w = -(1+z)/z$ is not. For the series we require $|z(1+z)(1+w)^2/w| < 1$. With $|z(1+z)| \le \varepsilon(1+\varepsilon)$ and $|w/(1+w)^2| \ge \gamma/(1+\gamma)^2$ we need $\varepsilon(1+\varepsilon) < \gamma/(1+\gamma)^2$. Observe that on $[0,1]$ we have $\gamma/(1+\gamma)^2 \ge \gamma/4$ since $4 \ge (1+\gamma)^2$. For $\gamma/4 > \varepsilon(1+\varepsilon)$ we choose $\gamma = 8\varepsilon$ with $\varepsilon \ll 1$ and we have our pair. Now for the pole at $-z/(1+z)$ we need for the maximum norm $\varepsilon/(1-\varepsilon) < \gamma = 8\varepsilon$ which holds with $\varepsilon < 7/8$ which we will enforce. The second pole under consideration is $-(1+z)/z = -1 - 1/z$. The closest this comes to the origin is $-1 + \varepsilon = -1 + \gamma/8$. To see that this is outside $|w| = \gamma$ we need $-1 + \gamma/8 < -\gamma$ or $\gamma < 8/9$. This means we instantiate

ε to $\varepsilon < 1/9$, which completes the discussion of the contour.

Now residues sum to zero and the residue at infinity in w is zero by inspection which means that the inner integral is minus the residue at $w = -(1+z)/z$, as it is equal to the sum of the residues at zero and at $w = -z/(1+z)$. We write

$$-\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\frac{1}{((1+z)/z+w)(z+(1+z)w)}\;dw.$$

We get from this being a simple pole the contribution (here \$(1+w)/w = 1/(1+z) \$)

$$-\frac{1}{z}\frac{1}{1+z}\frac{1}{z-(1+z)^2/z} = \frac{1}{1+z}\frac{1}{1+2z}$$

which combined with the integral in z gives

$$(-1)^n[z^n]\frac{1}{1+z}\frac{1}{1+2z} = (-1)^n\sum_{q=0}^{n}(-1)^q 2^q(-1)^{n-q} = \sum_{q=0}^{n} 2^q.$$

This is indeed

$$\boxed{2^{n+1}-1}$$

as claimed.

This was math.stackexchange.com problem 4196412.

58.10 Stirling numbers, Bernoulli numbers and Catalan numbers

Suppose we seek to prove that

$$\sum_{k=0}^{n}\left\{ n+k \atop k \right\}\binom{2n}{n+k}\frac{(-1)^k}{k+1} = B_n\binom{2n}{n}\frac{1}{n+1}$$

a unique identity that connects three types of significant combinatorial numbers. We get for the LHS

$$\sum_{k=0}^{n}\left\{ n+k \atop k \right\}\binom{2n}{n-k}\frac{(-1)^k}{k+1} = [z^n](1+z)^{2n}\sum_{k=0}^{n}\left\{ n+k \atop k \right\}\frac{(-1)^k z^k}{k+1}.$$

The coefficient extractor $[z^n]$ combined with the factor z^k enforces the upper limit of the sum so we may let k range to infinity:

$$[z^n](1+z)^{2n}\sum_{k\ge 0}(n+k)![w^{n+k}]\frac{(\exp(w)-1)^k}{k!}\frac{(-1)^k z^k}{k+1}.$$

Here we see that we will require complex methods and switch to

$$\frac{(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2n}}{z^{n+1}}$$
$$\times\sum_{k\ge 0}\binom{n+k}{n-1}(-1)^k z^k\frac{(\exp(w)-1)^k}{w^k}\;dz\;dw.$$

Computing the sum we find

$$-\frac{1}{z}\frac{w}{\exp(w)-1}\sum_{k\ge 1}\binom{n-1+k}{n-1}(-1)^k z^k\frac{(\exp(w)-1)^k}{w^k}$$
$$=\frac{1}{z}\frac{w}{\exp(w)-1}-\frac{1}{z}\frac{w}{\exp(w)-1}\frac{1}{(1+z(\exp(w)-1)/w)^n}.$$

The first component yields by inspection

$$(n-1)!\times\binom{2n}{n+1}\times B_n\frac{1}{n!}=B_n\frac{1}{n}\binom{2n}{n+1}=B_n\binom{2n}{n}\frac{1}{n+1}.$$

We have the claim if we can show the second component yields zero. We get

$$-\frac{(n-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{(\exp(w)-1)^{n+1}}$$
$$\times\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{2n}}{z^{n+2}}\frac{1}{(z+w/(\exp(w)-1))^n}\;dz\;dw.$$

At this time we must instantiate our contours. We need for the binomial series to converge that $|z(\exp(w)-1)/w|<1$ or $|z|<|w/(\exp(w)-1)|$. Observe that this means the pole at $z=-w/(\exp(w)-1)$ is outside the circle $|z|=\varepsilon$. To get a lower bound on the norm of the image of $|w|=\gamma$ we first take $\gamma\ll 1$ and observe that by expanding the series and bounding $|\sum_{m\ge 1}w^{m-1}/m!|$ by $\sum_{m\ge 1}\gamma^{m-1}/m!$ we have $|(\exp(w)-1)/w|\le(\exp(\gamma)-1)/\gamma$. Since the term in w is non-zero on and inside $|w|=\gamma$ (there is no pole at zero and the value there is one and the nearest zero is at $\pm 2\pi i$) we may invert to get $|w/(\exp(w)-1)|\ge\gamma/(\exp(\gamma)-1)$. Now we also have $\gamma/(\exp(\gamma)-1)>1-\frac{1}{2}\gamma$ as can be seen by comparing $\sum_{m\ge 1}\frac{1}{2^{m-1}}\gamma^m$ to $\sum_{m\ge 1}\frac{1}{m!}\gamma^m$, certainly both convergent for $\gamma\ll 1$. Hence $\varepsilon=1-\frac{1}{2}\gamma$ is an admissible choice and we have determined the contour. The pair $\gamma=1/3$ and $\varepsilon=5/6$ will work.

We thus must verify that the pole at $z=-w/(\exp(w)-1)$ makes a zero contribution (residues sum to zero and the residue at infinity is zero by inspection). This requires (Leibniz rule)

$$\frac{1}{(n-1)!}\left(\frac{1}{z^{n+2}}(1+z)^{2n}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^q \frac{(n+1+q)!}{(n+1)!} \frac{1}{z^{n+2+q}}$$

$$\times \frac{(2n)!}{(2n-(n-1-q))!} (1+z)^{2n-(n-1-q)}$$

$$= \sum_{q=0}^{n-1} (-1)^q \binom{n+1+q}{q} \frac{1}{z^{n+2+q}} \binom{2n}{n+1+q} (1+z)^{n+1+q}.$$

Observe that

$$\binom{n+1+q}{q}\binom{2n}{n+1+q} = \frac{(2n)!}{q! \times (n+1)! \times (n-1-q)!} = \binom{2n}{n+1}\binom{n-1}{q}$$

so the sum becomes

$$\binom{2n}{n+1} \frac{(1+z)^{n+1}}{z^{n+2}} \sum_{q=0}^{n-1} \binom{n-1}{q} (-1)^q \frac{(1+z)^q}{z^q}$$

$$= \binom{2n}{n+1} \frac{(1+z)^{n+1}}{z^{n+2}} \left(1 - \frac{1+z}{z}\right)^{n-1}$$

$$= \binom{2n}{n+1} (-1)^{n-1} \frac{(1+z)^{n+1}}{z^{2n+1}}.$$

Making the substitution we are left with the integral

$$\binom{2n}{n+1} (-1)^n \frac{(n-1)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{(\exp(w)-1)^{n+1}}$$

$$\times \frac{(1-w/(\exp(w)-1))^{n+1}}{(-w/(\exp(w)-1))^{2n+1}} \, dw.$$

The inner term is

$$-(\exp(w)-1)^n \frac{1}{w^{2n+1}} (1-w/(\exp(w)-1))^{n+1}$$

$$= -\frac{1}{w^{2n+1}} \frac{1}{\exp(w)-1} (\exp(w)-1-w)^{n+1}.$$

We get for the integral

$$\binom{2n}{n+1} (-1)^{n+1} \frac{(n-1)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{2n+2}} \frac{w}{\exp(w)-1} (\exp(w)-1-w)^{n+1} \, dw.$$

Now this is

$$[w^{2n+1}]\frac{w}{\exp(w)-1}(\exp(w)-1-w)^{n+1} = 0$$

because $(\exp(w)-1-w)^{n+1} = \frac{1}{2^{n+1}}w^{2n+2} + \cdots$ which concludes the argument. (The poles at $\pm 2\pi ik, k \ge 1$ are not inside the contour.)

This problem has not yet appeared at math.stackexchange.com. The source is exercise 6.74 from *Concrete Mathematics* by Graham, Knuth and Patashnik, [GKP89] credited to B.F.Logan.

58.11 Computing an EGF from an OGF

We seek to compute the EGF of a sequence from its OGF. There may be some cases where complex variables, the residue theorem and the residue at infinity are helpful. Suppose your OGF is $f(z)$ and the desired EGF is $g(w)$. Then we have

$$g(w) = \sum_{n\ge 0} \frac{w^n}{n!} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} f(z)\; dz.$$

This will simplify together with some conditions on convergence to give

$$g(w) = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z} \sum_{n\ge 0} \frac{1}{n!} \frac{w^n}{z^n}\; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z} \exp(w/z)\; dz.$$

Example I. Suppose

$$f(z) = \frac{1}{1-z},$$

which yields

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{1-z} \frac{1}{z} \exp(w/z)\; dz.$$

Now for $z = R\exp(i\theta)$ with R going to infinity we have

$$2\pi R \times \frac{1}{R^2} \times \exp(|w|/R) \to 0$$

as $R \to \infty$ so this integral is

$$-\mathrm{Res}_{z=1} \frac{1}{1-z} \frac{1}{z} \exp(w/z)$$

and we get

$$g(w) = \exp(w)$$

which is the correct answer.

Example II. This time suppose that

$$f(z) = \frac{z}{(1-z)^2}$$

so that we should get

$$g(w) = \sum_{n\ge 1} n\frac{w^n}{n!} = w\sum_{n\ge 1} \frac{w^{n-1}}{(n-1)!} = w\exp(w).$$

The integral formula yields

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z}{(1-z)^2}\frac{1}{z}\exp(w/z)\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)^2}\exp(w/z)\; dz.$$

The residue at infinity is zero as before and we have

$$\exp(w/z) = \sum_{n\ge 0} (\exp(w/z))^{(n)}\Big|_{z=1} \frac{(z-1)^n}{n!}$$

The coefficient on $(z-1)$ is

$$-\frac{1}{z^2} w\exp(w/z)\Big|_{z=1} = -w\exp(w)$$

which is the correct answer taking into account the sign flip due to $z = 1$ not being inside the contour.

Remark. Good news. The sum in the integral converges everywhere.

Addendum: somewhat more involved example. The OGF of Stirling numbers of the second kind for set partitions into k non-empty sets is

$$\sum_{n\ge 0} \left\{ n \atop k \right\} z^n = \prod_{q=1}^{k} \frac{z}{1-qz}.$$

We thus have that

$$g(w) = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\exp(w/z)\prod_{q=1}^{k}\frac{z}{1-qz}\; dz$$

$$= \frac{(-1)^k}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\exp(w/z)\prod_{q=1}^{k}\frac{z}{qz-1}\; dz$$

$$= \frac{(-1)^k}{k! \times 2\pi i} \int_{|z|=\epsilon} \frac{1}{z} \exp(w/z) \prod_{q=1}^{k} \frac{z}{z-1/q} \; dz.$$

Computing the sum of the residues at the finite poles not including zero we get

$$\frac{(-1)^k}{k!} \sum_{q=1}^{k} q \exp(qw) \times \frac{1}{q} \prod_{m=1}^{q-1} \frac{1/q}{1/q-1/m} \prod_{m=q+1}^{k} \frac{1/q}{1/q-1/m}$$

$$= \frac{(-1)^k}{k!} \sum_{q=1}^{k} \exp(qw) \prod_{m=1}^{q-1} \frac{m}{m-q} \prod_{m=q+1}^{k} \frac{m}{m-q}$$

$$= \frac{(-1)^k}{k!} \sum_{q=1}^{k} \exp(qw) \frac{k!}{q} \prod_{m=1}^{q-1} \frac{1}{m-q} \prod_{m=q+1}^{k} \frac{1}{m-q}$$

$$= \frac{(-1)^k}{k!} \sum_{q=1}^{k} \exp(qw) \frac{k!}{q} \frac{(-1)^{q-1}}{(q-1)!} \frac{1}{(k-q)!}$$

$$= -\frac{1}{k!} \sum_{q=1}^{k} \exp(qw)(-1)^{k-q} \binom{k}{q}$$

$$= -\left(\frac{(\exp(w)-1)^k}{k!} - \frac{(-1)^k}{k!} \right).$$

This is a case where the residue at infinity is not zero. We have the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

This yields for the present case

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z \exp(wz) \prod_{q=1}^{k} \frac{1/z}{1-q/z} = -\mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{z-q}$$

$$= -\frac{1}{k!} \mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{z/q-1}$$

$$= -\frac{(-1)^k}{k!} \mathrm{Res}_{z=0} \frac{1}{z} \exp(wz) \prod_{q=1}^{k} \frac{1}{1-z/q} = -\frac{(-1)^k}{k!}.$$

Adding the residue at infinity to the residues from the poles at $z = 1/q$ we finally obtain

$$-\left(\frac{(\exp(w)-1)^k}{k!}-\frac{(-1)^k}{k!}\right)-\frac{(-1)^k}{k!}=-\frac{(\exp(w)-1)^k}{k!}.$$

Taking into account the sign flip we have indeed computed the EGF of the Stirling numbers of the second kind

$$\sum_{n\ge 0}\left\{ {n \atop k} \right\}\frac{z^n}{n!}$$

as can be seen from the combinatorial class equation

$$\mathrm{SET}(\mathcal{U}\times\mathrm{SET}_{\ge 1}(\mathcal{Z}))$$

which gives the bivariate generating function

$$G(z,u)=\exp(u(\exp(z)-1)).$$

This was math.stackexchange.com problem 1289377.

58.12 Stirling numbers of the first and second kind

We seek an alternate closed form of

$$\sum_{q=0}^{r}(-1)^{q+r}\left[{r \atop q} \right]\left\{ {n+q-1 \atop k} \right\}.$$

With the usual EGFs this becomes

$$\sum_{q=0}^{r}(-1)^{q+r}\frac{r!}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{r+1}}\frac{1}{q!}\left(\log\frac{1}{1-z}\right)^q \\ \times\frac{(n+q-1)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+q}}\frac{1}{k!}(\exp(w)-1)^k\;dw\;dz.$$

Now we may extend q beyond r because $\left(\log\frac{1}{1-z}\right)^q=z^q+\cdots$ and hence $q>r$ produces no pole in a neighborhood of zero (the branch cut of the logarithmic term is $[1,\infty)$). We find

$$\frac{(-1)^r\times r!\times(n-1)!}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{r+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\frac{1}{k!}(\exp(w)-1)^k \\ \times\sum_{q\ge 0}\binom{n+q-1}{n-1}\frac{(-1)^q}{w^q}\left(\log\frac{1}{1-z}\right)^q\;dw\;dz.$$

Next we will sum the binomial series which requires $|\log \frac{1}{1-z}| < |w|$. Observe that for the image of $|z| = \varepsilon$ we have $|\log \frac{1}{1-z}| < \frac{\varepsilon}{1-\varepsilon}$. Therefore choosing γ so that $\frac{\varepsilon}{1-\varepsilon} \le \gamma$ will work e.g. for $\varepsilon = 1/Q$ we take $\gamma = 1/(Q-1)$. This yields

$$\frac{(-1)^r \times r! \times (n-1)!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} \frac{1}{k!} (\exp(w) - 1)^k \frac{1}{(1 + \frac{1}{w} \log \frac{1}{1-z})^n} \; dw \; dz$$
$$= \frac{(-1)^r \times r! \times (n-1)!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{k!} (\exp(w) - 1)^k \frac{1}{(w + \log \frac{1}{1-z})^n} \; dw \; dz.$$

The pole at zero for w has been canceled but the pole at $w = -\log \frac{1}{1-z}$ now lies inside the contour. Therefore we require

$$\frac{1}{(n-1)!} \left(\frac{1}{k!} (\exp(w) - 1)^k \right)^{(n-1)}$$
$$= \frac{1}{(n-1)! \times k!} \left(\sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} \exp(pw) \right)^{(n-1)}$$
$$= \frac{1}{(n-1)! \times k!} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} \exp(pw)$$

Evaluate at $w = -\log \frac{1}{1-z}$ and substitute into the integral in z to obtain

$$\frac{(-1)^r \times r!}{k! \times 2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} (1-z)^p \; dz$$
$$= (-1)^r \frac{r!}{k!} \sum_{p=r}^{k} \binom{k}{p} (-1)^{k-p} p^{n-1} (-1)^r \binom{p}{r}.$$

We have established that the sum vanishes when $k < r$. Note that

$$\binom{k}{p}\binom{p}{r} = \frac{k!}{(k-p)! \times r! \times (p-r)!} = \binom{k}{r}\binom{k-r}{p-r}$$

so this simplifies to

$$\frac{1}{(k-r)!} \sum_{p=r}^{k} \binom{k-r}{p-r} (-1)^{k-p} p^{n-1}.$$

We have proved that the alternate closed form is

$$\boxed{\frac{(-1)^{k-r}}{(k-r)!}\sum_{p=0}^{k-r}\binom{k-r}{p}(-1)^p(p+r)^{n-1}.}$$

An interesting special case is that this evaluates to r^{n-1} when $k = r$.

This problem has not appeared at math.stackexchange.com. It is from page 172 eqn. 12.22 of H.W.Gould's *Combinatorial Identities for Stirling Numbers* [Gou16] where it is attributed to Frank Olson.

58.13 An identity by Carlitz

We seek to show that where $m \ge 1$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{m} = \frac{n}{m}\binom{n-m-1}{m-1}2^{n-2m}.$$

We get for the LHS

$$[z^m]\sum_{k=0}^{n}\binom{n}{k}\sqrt{1+z}^k = [z^m](1+\sqrt{1+z})^n.$$

This is

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^{m+1}}(1+\sqrt{1+z})^n\; dz.$$

Now put $1+\sqrt{1+z} = w$ so that $z = w(w-2)$ and $dz = 2(w-1)\; dw$ to get

$$\frac{1}{2\pi i}\int_{|w-2|=\gamma}\frac{1}{w^{m+1}(w-2)^{m+1}}w^n 2(w-1)\; dw.$$

Now we have (series need not be finite)

$$\begin{aligned} w^{n-m} = (2+(w-2))^{n-m} &= 2^{n-m}(1+(w-2)/2)^{n-m} \\ &= 2^{n-m}\sum_{q\ge 0}\binom{n-m}{q}(w-2)^q/2^q \end{aligned}$$

so we get for the integral

$$\begin{aligned} &2^{n-m+1}\binom{n-m}{m}2^{-m} - 2^{n-m}\binom{n-m-1}{m}2^{-m} \\ &= 2^{n-2m}\binom{n-m-1}{m-1}\left[2\frac{n-m}{m} - \frac{n-2m}{m}\right] \\ &= 2^{n-2m}\binom{n-m-1}{m-1}\frac{n}{m}. \end{aligned}$$

This is the claim.

Remark. We need to document the choice of γ in terms of $\varepsilon \ll 1$. (The square root has the branch cut on $(-\infty, -1]$.) The image of $|z| = \varepsilon$ is contained in an annulus centered at two of radius $\sqrt{1+\varepsilon} - 1$ and $1 - \sqrt{1-\varepsilon}$. We may deform the image to a circle $|w-2| = \gamma$ where $\gamma = \epsilon/2$. This means the pole at $w = 0$ is definitely not inside the contour.

Using the residue operator

We get

$$\begin{aligned} & \operatorname*{res}_z \frac{1}{z^{m+1}}(1+\sqrt{1+z})^n \\ = & \operatorname*{res}_z \frac{1}{z^{m+1}}(-1)^n z^n \frac{1}{(1-\sqrt{1+z})^n}. \end{aligned}$$

Now we put $1 - \sqrt{1+z} = w$ so that $z = w(w-2)$ and $dz = 2(w-1)\, dw$ so that we obtain

$$\begin{aligned} & \operatorname*{res}_w \frac{1}{w^{m+1}(w-2)^{m+1}}(-1)^n w^n (w-2)^n \frac{1}{w^n} 2(w-1) \\ = & \operatorname*{res}_w \frac{1}{w^{m+1}(w-2)^{m-n+1}}(-1)^n 2(w-1) \\ = & \, 2^{n-m} \operatorname*{res}_w \frac{1}{w^{m+1}(w/2-1)^{m-n+1}}(-1)^n (w-1) \\ = & \, 2^{n-m}(-1)^{m+1} \operatorname*{res}_w \frac{1}{w^{m+1}(1-w/2)^{m-n+1}}(w-1). \end{aligned}$$

Extracting the residue yields

$$\begin{aligned} & 2^{n-m}(-1)^{m+1}\left(\binom{2m-n-1}{m-1}\frac{1}{2^{m-1}} - \binom{2m-n}{m}\frac{1}{2^m}\right) \\ = & \, 2^{n-m}(-1)^{m+1}\left((-1)^{m-1}\binom{n-m-1}{m-1}\frac{1}{2^{m-1}} - (-1)^m\binom{n-m-1}{m}\frac{1}{2^m}\right) \\ = & \, 2^{n-2m}\left(2\binom{n-m-1}{m-1} + \binom{n-m-1}{m}\right) \\ = & \, 2^{n-2m}\left(2\binom{n-m-1}{m-1} + \frac{n-2m}{m}\binom{n-m-1}{m-1}\right). \end{aligned}$$

Merge the two binomial coefficients to obtain the same answer as before.

This problem is from page 43 eqn. 3.163 of H.W.Gould's *Combinatorial Identities* [Gou72].

58.14 Logarithm squared of the Catalan number OGF

Suppose we seek to find

$$[z^n] \log\left(\frac{2}{1+\sqrt{1-4z}}\right).$$

This is given by

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \log\left(\frac{2}{1+\sqrt{1-4z}}\right)\; dz.$$

Now put $1-4z = w^2$ so that $z = 1/4(1-w^2)$ and $-2\; dz = w\; dw$ to get

$$\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{4^{n+1}}{(1-w^2)^{n+1}} \log\left(\frac{2}{1+w}\right)\left(-\frac{1}{2}\right) w\; dw.$$

This is

$$-\frac{1}{2}\frac{4^{n+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(1-w)^{n+1}}\frac{1}{(1+w)^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\times w\; dw$$

or

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{n+1}}\frac{1}{(1+w)^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\times w\; dw.$$

This has two parts, part A_1 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{n}}\frac{1}{(1+w)^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\; dw$$

and part A_2 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{n+1}}\frac{1}{(1+w)^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\; dw$$

Part A_1 is

$$\frac{1}{2}\frac{(-1)^n\times 4^{n+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{n}}\frac{1}{(2+(w-1))^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\; dw$$

$$= \frac{(-1)^n\times 2^{n}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{n}}\frac{1}{(1+(w-1)/2)^{n+1}} \log\left(\frac{1}{1+(w-1)/2}\right)\; dw.$$

Extracting coefficients we get

$$(-1)^n 2^n \sum_{q=0}^{n-2} \binom{q+n}{n} \frac{(-1)^q}{2^q} \frac{(-1)^{n-1-q}}{2^{n-1-q}\times(n-1-q)}$$

which is

$$-2\sum_{q=0}^{n-2}\binom{q+n}{n}\frac{1}{n-1-q}.$$

Part A_2 is

$$(-1)^n 2^n \sum_{q=0}^{n-1}\binom{q+n}{n}\frac{(-1)^q}{2^q}\frac{(-1)^{n-q}}{2^{n-q}\times(n-q)}$$

which is

$$\sum_{q=0}^{n-1}\binom{q+n}{n}\frac{1}{n-q}.$$

Re-index A_1 to match A_2, getting

$$-2\sum_{q=1}^{n-1}\binom{q-1+n}{n}\frac{1}{n-q}.$$

Collecting the two contributions we obtain

$$\frac{1}{n}+\sum_{q=1}^{n-1}\left(\binom{q+n}{n}-2\binom{q-1+n}{n}\right)\frac{1}{n-q}$$

which is

$$\frac{1}{n}+\sum_{q=1}^{n-1}\left(\frac{q+n}{q}\binom{q-1+n}{n}-2\binom{q-1+n}{n}\right)\frac{1}{n-q}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{n-q}{q}\binom{q-1+n}{n}\frac{1}{n-q}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{1}{q}\binom{q-1+n}{n}$$

$$=\frac{1}{n}+\sum_{q=1}^{n-1}\frac{(q-1+n)!}{q!\times n!}$$

$$=\frac{1}{n}+\frac{1}{n}\sum_{q=1}^{n-1}\frac{(q-1+n)!}{q!\times(n-1)!}$$

$$=\frac{1}{n}+\frac{1}{n}\sum_{q=1}^{n-1}\binom{q-1+n}{n-1}=\frac{1}{n}\sum_{q=0}^{n-1}\binom{q-1+n}{n-1}.$$

To evaluate this last sum we use the integral

$$\binom{n-1+q}{n-1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1+q}}{z^n}\; dz$$

which gives for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^n}\sum_{q=0}^{n-1}(1+z)^q\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^n}\frac{(1+z)^n-1}{1+z-1}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}}((1+z)^n-1)\; dz.$$

This also has two components, the second is zero and given by

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}}\; dz$$

leaving

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+1}}\; dz$$

which evaluates to

$$\binom{2n-1}{n}.$$

We have shown that

$$[z^n]\log\left(\frac{2}{1+\sqrt{1-4z}}\right) = \frac{1}{n}\binom{2n-1}{n}.$$

Addendum Feb 27 2022. It appears from the comments that OP wanted to prove

$$[z^n]\log^2\frac{2}{1+\sqrt{1-4z}} = \binom{2n}{n}(H_{2n-1}-H_n)\frac{1}{n}.$$

Using the result from the previous section the LHS becomes

$$\sum_{k=1}^{n-1}\frac{1}{k}\binom{2k-1}{k}\frac{1}{n-k}\binom{2n-2k-1}{n-k}.$$

Using

$$\frac{1}{k}\frac{1}{n-k} = \frac{1}{n}\frac{1}{k} + \frac{1}{n}\frac{1}{n-k}$$

this becomes

$$\frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{n-k}\binom{2k-1}{k}\binom{2n-2k-1}{n-k}$$
$$= \frac{1}{2n}\sum_{k=1}^{n-1}\frac{1}{n-k}\binom{2k}{k}\binom{2n-2k}{n-k}$$
$$= -\frac{1}{2n^2}\binom{2n}{n} + \frac{1}{2n}[w^n]\log\frac{1}{1-w}\sum_{k\ge 0} w^k\binom{2k}{k}\binom{2n-2k}{n-k}.$$

Here we have extended to infinity due to the coefficient extractor in w (note that $\log\frac{1}{1-w} = w + \cdots$) and canceled the value for $k=0$ that was included in the sum. Continuing with the inner sum term

$$[z^n]\frac{1}{\sqrt{1-4wz}}\frac{1}{\sqrt{1-4z}}$$
$$= [z^n]\frac{1}{\sqrt{(1-4z)^2 - 4z(1-4z)(w-1)}}$$
$$= [z^n]\frac{1}{1-4z}\frac{1}{\sqrt{1-4z(w-1)/(1-4z)}}$$
$$= [z^n]\sum_{k=0}^{n}\binom{2k}{k}z^k(w-1)^k\frac{1}{(1-4z)^{k+1}}.$$

This is

$$\frac{1}{2n}[w^n]\log\frac{1}{1-w}\sum_{k=0}^{n}\binom{2k}{k}(w-1)^k\binom{n}{k}4^{n-k}.$$

Recall from section **??** that with $1 \le k \le n$

$$\frac{1}{k} = \binom{n}{k}[w^n]\log\frac{1}{1-w}(w-1)^{n-k}.$$

Hence we get two pieces, the first is

$$\frac{1}{2n}\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{n-k}4^{n-k}.$$

and

$$\frac{1}{2n}[w^n]\log\frac{1}{1-w}\binom{2n}{n}(w-1)^n.$$

We get for the second

$$\binom{2n}{n}\frac{1}{2n}\operatorname*{res}_{w}\frac{1}{w^{n+1}}\log\frac{1}{1-w}(-1)^n(1-w)^n.$$

We put $w/(1-w)=v$ so that $w=v/(1+v)$ and $dw=1/(1+v)^2\; dv$ to get (without the scalar in front)

$$\operatorname*{res}_{v}\frac{1}{v^{n+1}}(1+v)\log\frac{1}{1-v/(1+v)}(-1)^n\frac{1}{(1+v)^2}$$
$$=\operatorname*{res}_{v}\frac{1}{v^{n+1}}(-1)^n\frac{1}{1+v}\log(1+v)=-(-1)^n[v^n]\frac{1}{1+v}\log\frac{1}{1+v}$$
$$=-[v^n]\frac{1}{1-v}\log\frac{1}{1-v}.$$

With the scalar we get

$$-\binom{2n}{n}\frac{1}{2n}H_n.$$

We have the result if we can show that the first piece is

$$\binom{2n}{n}\left(H_{2n-1}+\frac{1}{2n}-\frac{1}{2}H_n\right)\frac{1}{n}=\binom{2n}{n}\left(H_{2n}-\frac{1}{2}H_n\right)\frac{1}{n}$$

i.e.

$$F_n=\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{n-k}4^{n-k}=\binom{2n}{n}(2H_{2n}-H_n).$$

We have for the LHS

$$4^n[w^n]\log\frac{1}{1-w}\sum_{k=0}^{n-1}\binom{2k}{k}w^k4^{-k}.$$

The coefficient extractor enforces the upper limit, we may extend to infinity and we find

$$4^n[w^n]\log\frac{1}{1-w}\frac{1}{\sqrt{1-w}}=[w^n]\log\frac{1}{1-4w}\frac{1}{\sqrt{1-4w}}.$$

Call the OGF $F(w)$. We get

$$F'(w)=\frac{4}{\sqrt{1-4w}^3}+\frac{2}{1-4w}F(w).$$

Extracting the coefficient on $[w^n]$ we get

$$(n+1)F_{n+1}=4^{n+1}(-1)^n\binom{-3/2}{n}+2\sum_{q=0}^{n}F_q4^{n-q}$$

$$= 4^{n+1}(-1)^n \frac{n+1}{(-1/2)} \binom{-1/2}{n+1} + 2 \sum_{q=0}^{n} F_q 4^{n-q}$$

$$= 2(n+1)\binom{2n+2}{n+1} + 2 \sum_{q=0}^{n} F_q 4^{n-q}$$

which also yields

$$\frac{1}{4}(n+2)F_{n+2} = \frac{1}{2}(n+2)\binom{2n+4}{n+2} + 2 \sum_{q=0}^{n+1} F_q 4^{n-q}.$$

Subtract to get

$$\frac{1}{4}(n+2)F_{n+2}$$

$$= (n+1)F_{n+1} + \frac{1}{2}(n+2)\binom{2n+4}{n+2} - 2(n+1)\binom{2n+2}{n+1} + \frac{1}{2}F_{n+1}.$$

Introducing $G_n = F_n \binom{2n}{n}^{-1}$ and dividing by $\binom{2n+2}{n+1}$ we get

$$\frac{1}{2}(2n+3)G_{n+2} = (n+3/2)G_{n+1} + 1 \quad \text{or} \quad G_n = G_{n-1} + \frac{1}{n-1/2}.$$

so that

$$G_n = \sum_{q=1}^{n} \frac{1}{q-1/2} = 2\sum_{q=1}^{n} \frac{1}{2q-1} = 2H_{2n-1} - H_{n-1} = 2H_{2n} - H_n.$$

This is the claim (we have $F_0 = G_0 = 0$ from the generating function) and it completes the entire argument.

This was math.stackexchange.com problem 1148203.

58.15 Bernoulli / Stirling number identity

We seek to show that

$$\sum_{k=0}^{n} \left[{n+1 \atop k+1}\right] B_k = \frac{n!}{n+1}.$$

The LHS is

$$\sum_{k=0}^{n} \left[{n+1 \atop n-k+1}\right] B_{n-k}$$

$$= \sum_{k=0}^{n} \left[{n+1 \atop n-k+1} \right] (n-k)! [z^{n-k}] \frac{z}{\exp(z)-1}$$

$$= [z^n] \frac{z}{\exp(z)-1} \sum_{k=0}^{n} (n-k)! z^k (n+1)! [w^{n+1}] \frac{1}{(n-k+1)!} \left(\log \frac{1}{1-w} \right)^{n-k+1}$$

$$= (n+1)! [z^n][w^{n+1}] \frac{z}{\exp(z)-1} \sum_{k=0}^{n} \frac{z^k}{n-k+1} \left(\log \frac{1}{1-w} \right)^{n-k+1}$$

$$= n! [z^n][w^n] \frac{z}{\exp(z)-1} \frac{1}{1-w} \sum_{k=0}^{n} z^k \left(\log \frac{1}{1-w} \right)^{n-k}.$$

We can certainly extend the sum to infinity because of the coefficient extractor in z and we find

$$n! [z^n][w^n] \frac{z}{\exp(z)-1} \frac{1}{1-w} \left(\log \frac{1}{1-w} \right)^{n} \frac{1}{1 - z/\log \frac{1}{1-w}}.$$

At this point we recognize that we need complex variables and write

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \left(\log \frac{1}{1-w} \right)^{n}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \frac{z}{\exp(z)-1} \frac{1}{1 - z/\log \frac{1}{1-w}} \; dz \; dw.$$

Now when we summed the geometric series we introduced the condition $|z| < \left| \log \frac{1}{1-w} \right|$ for convergence. Continuing,

$$-\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} \frac{1}{1-w} \left(\log \frac{1}{1-w} \right)^{n+1}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} \frac{z}{\exp(z)-1} \frac{1}{z - \log \frac{1}{1-w}} \; dz \; dw.$$

Now we have with $\gamma \ll 1$ that

$$\left| \log \frac{1}{1-w} \right| > \gamma - \sum_{q \ge 2} \frac{\gamma^q}{q} > \gamma - \sum_{q \ge 2} \gamma^q = \gamma - \frac{\gamma^2}{1-\gamma}.$$

This is because $\log \frac{1}{1-w} = w + w^2/2 + w^3/3 + \cdots$ and the first term has modulus γ so that we minimize the whole if we subtract the maximum modulus of all remaining terms. If we choose this last term for ε we have convergence. As an example $\gamma = 1/10$ and $\varepsilon = 4/45$ will work.

Summarizing we have that all the poles in z at $\rho_k = 2\pi i k$ where $|k| \ge 1$ are outside the contour in z, as is the pole at $\log \frac{1}{1-w}$. Note also that with the principal branch of the logarithm $|\arg(\log \frac{1}{1-w})| \le \pi$ so the pole at $\log \frac{1}{1-w}$ does not coincide with any of the ρ_k.

We evaluate the inner integral using the fact that residues sum to zero. Hence we require minus the contribution from the logarithm pole and minus the contribution from the ρ_k. The former yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\left(\log\frac{1}{1-w}\right)^{-n}\frac{1}{1/(1-w)-1}\,dw$$
$$=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+2}}\log\frac{1}{1-w}\,dw=\frac{1}{n+1}.$$

Multiply by $n!$ to get $n!/(n+1)$ which is the claim. It remains to show that the ρ_k contribute zero. These are all simple. We get

$$\lim_{z\to\rho_k}\frac{z-\rho_k}{\exp(z)-1}=\lim_{z\to\rho_k}\frac{1}{\exp(z)}=1.$$

This yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\frac{1}{\rho_k^n}\frac{1}{\rho_k-\log\frac{1}{1-w}}\,dw.$$

Note that

$$\left|\log\frac{1}{1-w}\right|=\left|\sum_{q\ge 1}\frac{w^q}{q}\right|\le\sum_{q\ge 1}\frac{\gamma^q}{q}<\frac{\gamma}{1-\gamma}.$$

With the choice of contour above we have a bound of $1/9$ on this term. Hence we may expand into a convergent geometric series, of which only the initial segment can possibly contribute:

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\frac{1}{\rho_k^{n+1}}\frac{1}{1-\frac{1}{\rho_k}\log\frac{1}{1-w}}\,dw$$
$$=[w^n]\frac{1}{1-w}\left(\log\frac{1}{1-w}\right)^{n+1}\frac{1}{\rho_k^n}\sum_{q=0}^{n}\frac{1}{\rho_k^q}\left(\log\frac{1}{1-w}\right)^q.$$

Note however that we are extracting a coefficient on $[w^n]$ from a term that starts at w^{n+1+q}, and we have a zero contribution, which concludes the argument.

This was DLMF [DLMF, Eq. 24.15.8].

58.16 Formal power series vs contour integration

We seek to show that

$$\sum_{q=0}^K (-1)^q \binom{2n+1-q}{q}\binom{2n-2q}{K-q} = \frac{1}{2}(1+(-1)^K).$$

We do it two ways, using formal power series and residue operators as well as contour integration.

Formal power series

The LHS is

$$\sum_{q=0}^K (-1)^q \binom{2n+1-q}{2n+1-2q}\binom{2n-2q}{K-q}$$
$$= [z^K](1+z)^{2n} \sum_{q=0}^K (-1)^q \binom{2n+1-q}{2n+1-2q} \frac{z^q}{(1+z)^{2q}}.$$

Here we may extend to infinity because of the coefficient extractor in z. We find

$$[z^K](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1} \sum_{q\ge 0} (-1)^q \frac{z^q}{(1+z)^{2q}} \frac{w^{2q}}{(1+w)^q}$$
$$= [z^K](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1} \frac{1}{1+zw^2/(1+z)^2/(1+w)}$$
$$= [z^K](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2} \frac{1}{(1+z)^2(1+w)+zw^2}.$$

Important note: what we have here is that $\binom{2n+1-q}{2n+1-2q}$ is zero when $2n+1-2q$ goes negative. This is not always what CAS systems might use. Maple for example uses that $\binom{a}{b}$ for $b<0$ is $\binom{a}{a-b}$ if $a \ge b$. This is a generalization that we will use in the second answer. It applies here because we replace $\binom{2n+1-q}{q}$ by $\binom{2n+1-q}{2n+1-2q}$ which is zero when $2q > 2n+1$ under the first rule.

The contribution from w is

$$\operatorname*{res}_w \frac{1}{w^{2n+2}}(1+w)^{2n+2} \frac{1}{(1+z)^2(1+w)+zw^2}.$$

Now we put $w/(1+w) = v$ so that $w = v/(1-v)$ and $dw = \frac{1}{(1-v)^2}\, dv$ to get

$$\operatorname*{res}_v \frac{1}{v^{2n+2}} \frac{1}{(1+z)^2(1+v/(1-v))+zv^2/(1-v)^2} \frac{1}{(1-v)^2}.$$

Restoring the coefficient extractor in z we have

$$[z^K](1+z)^{2n+2}[v^{2n+1}] \frac{1}{(1+z)^2(1-v)+zv^2}$$

$$= [z^K](1+z)[v^{2n+1}]\frac{1}{(1+z)^2-(1+z)^3v+zv^2(1+z)^2}$$

$$= [z^K]\frac{1}{1+z}[v^{2n+1}]\frac{1}{1-(1+z)v+zv^2}$$

$$= [z^K]\frac{1}{1+z}[v^{2n+1}]\frac{1}{(1-v)(1-vz)}$$

$$= [z^K]\frac{1}{1+z}\sum_{q=0}^{2n+1} z^q = [z^K]\frac{1-z^{2n+2}}{1-z^2}.$$

This is clearly an even function hence zero when K is odd. When K is even and $K < 2n+2$ we get a value of one, and when K is even and $K \ge 2n+2$ the two contributions cancel, for a value of zero. Thus we have

$$\frac{1}{2}(1+(-1)^K)[[K < 2n+2]].$$

Contour integration

We again seek to show that

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q} = \frac{1}{2}(1+(-1)^K).$$

This time we will not flip the lower index of the first binomial coefficient and we get an answer that agrees with the second rule, which is used by CAS.

The LHS is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\sum_{q=0}^{K}(-1)^q\frac{1}{z^{q+1}}(1+z)^{2n+1-q}\frac{1}{w^{K-q+1}}(1+w)^{2n-2q}\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$

$$\times\sum_{q=0}^{K}(-1)^q\frac{1}{z^q}(1+z)^{-q}w^q(1+w)^{-2q}\;dz\;dw.$$

Here we may extend q beyond K to infinity because the pole at zero in w is canceled for the extra values. We obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1+w)^{2n}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}$$

$$\times\frac{1}{1+w/(1+w)^2/z/(1+z)}\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{K+1}}(1+w)^{2n+2}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$
$$\times\frac{1}{(1+z(1+w))(w+z(1+w))}\; dz\; dw.$$

The pole at zero in z is gone but a new pole has appeared inside the contour. Note that when we summed the geometric series we required $|w/(1+w)^2| < |z(1+z)|$. We have with $\gamma \ll 1$ and $\varepsilon \ll 1$ that $|w/(1+w)^2| \le \gamma/(1-\gamma)^2 < 2\gamma$ and $|z(1+z)| \ge \varepsilon(1-\varepsilon) > \frac{1}{2}\varepsilon$. Therefore taking $\varepsilon = 4\gamma$ will work e.g. $\gamma = 1/11$ and $\varepsilon = 4/11$.

We have for the first simple pole at $z_0 = -1/(1+w)$ that $|-1/(1+w)| > 1/(1+\gamma) > 4\gamma = \varepsilon$. This pole is not inside the contour. The second pole is at $z_1 = -w/(1+w)$ and we have $|-w/(1+w)| < \gamma/(1-\gamma) < 4\gamma = \varepsilon$. This pole is inside the contour. We thus write

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{K+1}}(1+w)^{2n+1}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}$$
$$\times\frac{1}{(1+z(1+w))(w/(1+w)+z)}\; dz\; dw.$$

Evaluating the residue from the simple pole at z_1 we find

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{K+1}}(1+w)^{2n+1}(1-w/(1+w))^{2n+2}\frac{1}{1-(1+w)w/(1+w)}\; dw$$
$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{K+1}}\frac{1}{1-w^2}\; dw.$$

This is

$$[w^K]\frac{1}{1-w^2} = \frac{1}{2}(1+(-1)^K)$$

as claimed.

Alternate evaluation

Returning to the start we seek

$$\sum_{q=0}^{K}(-1)^q\binom{2n+1-q}{q}\binom{2n-2q}{K-q} = \frac{1}{2}(1+(-1)^K).$$

With

$$\binom{2n-2q}{K-q} = \binom{2n-2q}{2n-K-q} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K-q+1}}\frac{1}{(1-w)^{2n-K-q+1}}\; dw$$

the LHS becomes

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\sum_{q=0}^{K}(-1)^q\frac{1}{z^{q+1}}(1+z)^{2n+1-q}\frac{1}{w^{K-q+1}}\frac{1}{(1-w)^{2n-K-q+1}}\;dz\;dw.$$

Here we may extend the sum to infinity due to the pole at $w=0$ vanishing when $q>K$. This yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}\\
&\qquad\times\sum_{q\geq 0}(-1)^q\frac{1}{z^q}(1+z)^{-q}w^q(1-w)^q\;dz\;dw\\
&=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z}(1+z)^{2n+1}\\
&\qquad\times\frac{1}{1+w(1-w)/z/(1+z)}\;dz\;dw\\
&=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}\\
&\qquad\times\frac{1}{z(1+z)+w(1-w)}\;dz\;dw\\
&=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}\frac{1}{2\pi i}\int_{|z|=\varepsilon}(1+z)^{2n+2}\\
&\qquad\times\frac{1}{(z+w)(z-(w-1))}\;dz\;dw.
\end{aligned}$$

Once more the pole at $z=0$ is gone but a new one has appeared (two of them, in fact). To see this note that in the summation of the infinite series we require for convergence that $|w(1-w)|<|z(1+z)|$. We have for $\varepsilon \ll 1$ and $\gamma \ll 1$ that $|z(1+z)|\geq \varepsilon(1-\varepsilon)\geq \frac{1}{2}\varepsilon$ and $|w(1-w)|\leq \gamma(1+\gamma)\leq \frac{3}{2}\gamma$. Hence $3\gamma<\varepsilon$ will work e.g. take $\gamma=\varepsilon/4$ as in $\gamma=1/16$ and $\varepsilon=1/4$. In particular $|w|<|z|$ so the pole at $z_0=-w$ is inside the contour. On the other hand the closest that the pole at $z_1=w-1$ which is on a circle at $z=-1$, rotating with radius γ, gets to the origin is $1-\gamma=1-\varepsilon/4>\varepsilon$ as long as $\varepsilon<4/5$, so this is definitely not inside the contour. Hence the contribution from $z_0=-w$ is the only one and it yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}\frac{1}{(1-w)^{2n-K+1}}(1-w)^{2n+2}\frac{1}{1-2w}\;dw\\
&=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{K+1}}(1-w)^{K+1}\frac{1}{1-2w}\;dw
\end{aligned}$$

$$= \sum_{q=0}^{K} \binom{K+1}{q} (-1)^q 2^{K-q} = -(-1)^{K+1}\frac{1}{2} + \frac{1}{2}\sum_{q=0}^{K+1} \binom{K+1}{q} (-1)^q 2^{K+1-q}$$

$$= -(-1)^{K+1}\frac{1}{2} + \frac{1}{2}1^{K+1} = \frac{1}{2}(1+(-1)^K).$$

We once more have the claim.

This was math.stackexchange.com problem 4597569.

58.17 A sum of the derivatives of inverse powers

We seek to show that

$$\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \left[\frac{1}{f^k(x)}\right]^{(n)} f^{k+1}(x) = [f(x)]^{(n)}.$$

Suppose for the moment that x is a complex number. We will assume $f(x)$ and $1/f(x)$ are analytic at x so they both have power series there with some non-zero radius of convergence. Take $\rho \ll$ the lesser of the two and take $\rho = 1$ if both converge everywhere. We then get for the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} f^{k+1}(x) \frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{1}{(z-x)^{n+1}} \frac{1}{f^k(z)}\; dz.$$

This is

$$\frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{f(z)}{(z-x)^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} f^{k+1}(x) \frac{1}{f^{k+1}(z)}\; dz$$

$$= \frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{f(z)}{(z-x)^{n+1}}\; dz$$

$$+ \frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{f(z)}{(z-x)^{n+1}} \sum_{k=-1}^{n} (-1)^k \binom{n+1}{k+1} f^{k+1}(x) \frac{1}{f^{k+1}(z)}\; dz$$

$$= f^{(n)}(x) - \frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{f(z)}{(z-x)^{n+1}} \left[1 - \frac{f(x)}{f(z)}\right]^{n+1}\; dz$$

$$= f^{(n)}(x) - \frac{n!}{2\pi i} \int_{|z-x|=\rho} \frac{1}{f^n(z)} \left[\frac{f(z)-f(x)}{z-x}\right]^{n+1}\; dz = f^{(n)}(x).$$

Expanding the bracketed term into a power series about x we find

$$g(z) = f'(x) + f''(x)(z-x)/2 + f'''(x)(z-x)^2/6 + \cdots$$

so it goes to a constant $f'(x)$ with no more pole at $z = x$. More importantly it is convergent in the neighborhood of x so it represents an analytic function there. Integrating $g^{n+1}(z)/f^n(z)$ which is the product of two analytic terms we obtain a zero contribution from the integral which concludes the argument.

This was math.stackexchange.com problem 4993025.

58.18 Shifted central binomial coefficient convolution

We seek to simplify

$$\sum_{k=0}^{n}(-1)^k\binom{k+r}{r}\binom{2k+2r}{k+r}\binom{n-k+r}{r}\binom{2n-2k+2r}{n-k+r}.$$

With this in mind we introduce

$$F_r(z) = \sum_{k\ge 0}\binom{k+r}{r}\binom{2k+2r}{k+r}z^k.$$

For a closed form we have

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{r+1}}(1+w)^{2r}\sum_{k\ge 0}\binom{k+r}{r}\frac{(1+w)^{2k}}{w^k}z^k\;dw.$$

For the series to converge we need $|(1+w)^2z/w| < 1$ or $|z| < |w/(1+w)^2|$. We will take $\gamma = 1$ which gives validity for $|z| < 1/4$. Continuing,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{r+1}}(1+w)^{2r}\frac{1}{(1-(1+w)^2z/w)^{r+1}}\;dw\\
&= \frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2r}\frac{1}{(w-(1+w)^2z)^{r+1}}\;dw\\
&= (-1)^{r+1}\frac{1}{z^{r+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2r}\frac{1}{((1+w)^2-w/z)^{r+1}}\;dw.
\end{aligned}$$

The fraction under the integral factors into $(w-\rho_0)(w-\rho_1)$ where $\rho_{0,1} = (1-2z\pm\sqrt{1-4z})/2/z$. We have with $|z| < 1/4$ the convergent series expansion

$$\rho_1 = -\frac{1}{2z}\sum_{k\ge 2}z^k[v^k]\sqrt{1-4v}.$$

Note that we have $|1/z| > 4$. Now the coefficients of the square root are all negative except for the constant term which is not included. Therefore the norm of the sum term is at most

$$\frac{1}{2|z|}\left(1-2|z|-\sqrt{1-4|z|}\right) < 1.$$

This is because $1-2|z|-\sqrt{1-4|z|} < 2|z|$ or $1-4|z| < \sqrt{1-4|z|}$ which holds when $1-4|z| < 1$ ($z = 0$ is not included here) and we find that ρ_1 is inside the contour (the unit circle). From the quadratic $\rho_0 = 1/\rho_1$ so ρ_0 is not. We then have from the integral evaluating at $w = \rho_1$ where we require

$$\frac{1}{r!}\left(\frac{1}{(w-\rho_0)^{r+1}}(1+w)^{2r}\right)^{(r)}$$

which is with the Leibniz rule

$$\frac{1}{r!}\sum_{q=0}^{r}\binom{r}{q}\frac{(-1)^q(r+1)^{\overline{q}}}{(w-\rho_0)^{r+1+q}}(1+w)^{2r-(r-q)}(2r)^{\underline{r-q}}$$
$$=\sum_{q=0}^{r}\frac{(-1)^q}{(w-\rho_0)^{r+1+q}}\binom{r+q}{q}(1+w)^{r+q}\binom{2r}{r-q}.$$

Observe that

$$\binom{r+q}{q}\binom{2r}{r-q} = \frac{(2r)!}{(r-q)!\times r!\times q!} = \binom{2r}{r}\binom{r}{q}$$

so that we find

$$\binom{2r}{r}\frac{1}{(w-\rho_0)^{r+1}}(1+w)^r\sum_{q=0}^{r}\binom{r}{q}(-1)^q\frac{(1+w)^q}{(w-\rho_0)^q}$$
$$=\binom{2r}{r}\frac{1}{(w-\rho_0)^{r+1}}(1+w)^r\left[1-\frac{1+w}{w-\rho_0}\right]^r$$
$$=\binom{2r}{r}\frac{1}{(w-\rho_0)^{2r+1}}(1+w)^r(-1)^r(1+\rho_0)^r.$$

Evaluating at $w = \rho_1$ we have $(1+w)(1+\rho_0) = 2+\rho_0+\rho_1 = 2+(2-4z)/2/z = \frac{1}{z}$ to get collecting everything

$$F_r(z) = (-1)^{r+1}\frac{1}{z^{r+1}}\binom{2r}{r}(-1)^{2r+1}\frac{z^{2r+1}}{\sqrt{1-4z}^{2r+1}}\frac{1}{z^r}(-1)^r = \binom{2r}{r}\frac{1}{(1-4z)^{r+1/2}}.$$

We may now evaluate the sum almost by inspection and get

$$\binom{2r}{r}^2[z^n]\frac{1}{(1+4z)^{r+1/2}}\frac{1}{(1-4z)^{r+1/2}} = \binom{2r}{r}^2[z^n]\frac{1}{(1-16z^2)^{r+1/2}}.$$

We see that n must be even in which case we obtain

$$\binom{2r}{r}^2[z^{n/2}]\frac{1}{(1-16z)^{r+1/2}} = \binom{2r}{r}^2(-1)^{n/2}16^{n/2}\binom{-r-1/2}{n/2}.$$

We can re-write the second binomial coefficient somewhat,

$$\frac{1}{(n/2)!}\prod_{p=0}^{n/2-1}(-r-1/2-p) = \frac{1}{(n/2)!}\frac{(-1)^{n/2}}{2^{n/2}}\prod_{p=0}^{n/2-1}(2r+1+2p)$$

$$= \frac{1}{(n/2)!}\frac{(-1)^{n/2}}{2^{n/2}}\frac{(2r+n-1)!!}{(2r-1)!!} = \frac{1}{(n/2)!}\frac{(-1)^{n/2}}{2^{n/2}}\frac{r!2^r}{(2r)!}\frac{(2r+n)!}{(r+n/2)!2^{r+n/2}}.$$

Plug this into the closed form to get

$$\binom{2r}{r}^2 2^n \binom{2r}{r}^{-1}\binom{2r+n}{r}\binom{r+n}{n/2} = 2^n\binom{2r}{r}\binom{2r+n}{r}\binom{r+n}{n/2}.$$

Remark. On seeing that $F_r(z)$ has such a simple form we gather that it can be computed very straightforwardly from first principles. We note that

$$F_r(z) = \frac{1}{r!}\left(\sum_{k\ge 0}\binom{2k+2r}{k+r}z^{k+r}\right)^{(r)}$$

$$= \frac{1}{r!}\left(\sum_{k\ge 0}\binom{2k}{k}z^k - \sum_{k=0}^{r-1}\binom{2k}{k}z^k\right)^{(r)}$$

$$= \frac{1}{r!}\left(\frac{1}{\sqrt{1-4z}}\right)^{(r)} = \frac{1}{r!}(1-4z)^{-1/2-r}(-1/2)^{\underline{r}}(-4)^r$$

$$= \frac{1}{(1-4z)^{r+1/2}}\binom{-1/2}{r}(-4)^r = \frac{1}{(1-4z)^{r+1/2}}[w^r]\frac{1}{\sqrt{1-4w}}$$

$$= \binom{2r}{r}\frac{1}{(1-4z)^{r+1/2}}.$$

This identity was from a paper by Lucio Barabesi in the JIS, the full citation is [Bar23].

58.19 Stirling cycle numbers and an exponential EGF

We seek to verify that

$$\left[{n \atop m}\right] = (-1)^{n-m}\binom{n-1}{m-1}(n-m)![z^{n-m}]\left(\frac{z}{\exp(z)-1}\right)^n.$$

We may simplify the factorial,

$$(-1)^{n-m}\frac{(n-1)!}{(m-1)!}[z^{n-m}]\left(\frac{z}{\exp(z)-1}\right)^n.$$

The contribution from z is

$$\operatorname*{res}_{z} \frac{1}{z^{n-m+1}} \left(\frac{z}{\exp(z)-1}\right)^n = \operatorname*{res}_{z} z^{m-1} \frac{1}{(\exp(z)-1)^n}.$$

Now put $z = \log(1+w)$ so that $dz = \frac{1}{1+w}\, dw$ to get

$$\operatorname*{res}_{w} \log^{m-1}(1+w) \frac{1}{w^n} \frac{1}{1+w}$$

$$= [w^{n-1}] \log^{m-1}(1+w) \frac{1}{1+w} = \frac{n}{m} [w^n] \log^m(1+w).$$

Restoring the factors in front,

$$(-1)^{n-m} \frac{n!}{m!} [w^n] \log^m(1+w) = (-1)^m \frac{n!}{m!} [w^n] \log^m(1-w)$$

$$= \frac{n!}{m!} [w^n] (-1)^m \log^m(1-w) = n! [w^n] \frac{1}{m!} \left(\log \frac{1}{1-w}\right)^m.$$

We have obtained the standard Stirling cycle number EGF and may conclude. This was Formal Laurent Series. It can also be done using complex variables. Noting that $z/(\exp(z)-1)$ is analytic in a neighborhood of the origin we obtain from the Cauchy Coefficient Formula

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n-m+1}} \left(\frac{z}{\exp(z)-1}\right)^n \, dz.$$

The substitution $z = \log(1+w)$ then maps $z = 0$ to $w = 0$ and is locally invertible. With $w = \exp(z) - 1$ we have $w = z + \sum_{q \geq 2} \frac{z^q}{q!}$. The first term produces a circle with the same radius as the image of $|z| = \varepsilon$. The sum of the remaining terms is bounded by $|z|^2/(1-|z|)$. Now put $\varepsilon = 1/Q$, with Q large. The bound on the remainder becomes $1/Q/(Q-1)$ which shows that it is of lower order than the circle term and we may deform the image contour to $|w| = \gamma$ where $\gamma < 1/Q - 1/Q/(Q-1)$, getting the integral

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \log^{m-1}(1+w) \frac{1}{w^n} \frac{1}{1+w} \, dw.$$

We may then apply the CCF one more time to return to the realm of formal power series. The above also shows that the singularity at $w = -1$ and the branch cut do not intersect the initial image contour (before shrinking to $|w| = \gamma$) given that it is contained in a circle of radius $1/Q + 1/Q/(Q-1) = 1/(Q-1) \ll 1$.

This was math.stackexchange.com problem 5046903.

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