Set partitions into labeled 3-tori, inverse transform

Marko Riedel

March 2026

The sequence OEIS A393519 documents set partitions into labeled 3-tori. For brevity we shall use the term torus numbers to refer to set partitions into q-tori. The cited sequence refers to the combinatorial class $\mathcal{T} = \textsc{Set}(\textsc{Tor}_{3,\ge 1}(\mathcal{Z}))$ which is partitions of $[n]$ into sets of 3-tori. Just as permutations are set partitions into cycles with class $\textsc{Set}(\textsc{Cyc}_{\ge 1}(\mathcal{Z}))$ and counted by Stirling cycle numbers we investigate the next to next dimension, where cycles become tori and tori become 3-tori. Torus numbers describe paritions into q-tori which we shall denote

$$\left|\begin{matrix} n \\ k \end{matrix}\right|_q.$$

A table is given at the cited sequence. We then have

$$\left|\begin{matrix} n \\ k \end{matrix}\right|_1 = \left[\begin{matrix} n \\ k \end{matrix}\right].$$

Recalling the Stirling number transform

This transform of a sequence $\{a_n\}$ is given by

$$b_n = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix}\right] (-1)^{n-k} a_k.$$

It has the important property that the inverse transform is given by

$$a_n = \sum_{k=0}^n \left\{\begin{matrix} n \\ k \end{matrix}\right\} b_k.$$

To see this substitute into the above to get

$$\begin{aligned} \sum_{k=0}^n \left\{\begin{matrix} n \\ k \end{matrix}\right\} b_k &= \sum_{k=0}^n \left\{\begin{matrix} n \\ k \end{matrix}\right\} \sum_{q=0}^k \left[\begin{matrix} k \\ q \end{matrix}\right] (-1)^{k-q} a_q \\ &= \sum_{q=0}^n a_q (-1)^q \sum_{k=q}^n \left\{\begin{matrix} n \\ k \end{matrix}\right\} \left[\begin{matrix} k \\ q \end{matrix}\right] (-1)^k \\ &= \sum_{q=0}^n a_q (-1)^q \sum_{k=q}^n \left\{\begin{matrix} n \\ k \end{matrix}\right\} k! [z^k] \frac{1}{q!} \left(\log \frac{1}{1-z}\right)^q (-1)^k \\ &= n! [w^n] \sum_{q=0}^n a_q (-1)^q \sum_{k=q}^n (\exp(w) - 1)^k (-1)^k [z^k] \frac{1}{q!} \left(\log \frac{1}{1-z}\right)^q. \end{aligned}$$

We may extend k to infinity since $\exp(w) - 1 = w + \cdots$ is under the extractor in w. Continuing,

$$\begin{aligned} n! [w^n] \sum_{q=0}^n a_q (-1)^q \frac{1}{q!} \sum_{k \ge q} (\exp(w) - 1)^k (-1)^k [z^k] \left(\log \frac{1}{1-z}\right)^q \\ = n! [w^n] \sum_{q=0}^n a_q (-1)^q \frac{1}{q!} \left(\log \frac{1}{1 - (1 - \exp(w))}\right)^q \\ = n! [w^n] \sum_{q=0}^n a_q (-1)^q (-w)^q \frac{1}{q!} = n! a_n (-1)^n (-1)^n \frac{1}{n!} = a_n. \end{aligned}$$

With this computation in mind we can now ask an interesting question, namely what is the inverse transform of the torus number transform:

$$b_n = \sum_{k=0}^n \left|\begin{matrix} n \\ k \end{matrix}\right|_q (-1)^{n-k} a_k.$$

Inverting the transform

From the cited sequence we have that the logarithmic term from the Stirling cycle numbers is now replaced by the more general term

$$\sum_{m\ge 1} \tau_q(m) \frac{z^m}{m}$$

which produces the logarithm when $q = 1$ (consult sequence). By a process completely analogous to the preceding computation we require a function $G(w)$ such that

$$\sum_{m\ge 1} \tau_q(m) \frac{(-1)^{m+1} G(w)^m}{m} = w.$$

This can be done by Lagrange inversion consult e.g. *The method of coefficients* by Merlini, Sprugnoli, and Verri. In seeking the inverse of

$$f(t) = \sum_{m\ge 1} \tau_q(m) \frac{(-1)^{m+1} t^m}{m}$$

we obtain that

$$[w^n] G(w) = \frac{1}{n} [t^{n-1}] \left(\frac{t}{f(t)}\right)^n.$$

This lets us compute the expansion of $G(w)$, which is an exponential generating function. We find

$$G(w) = w + \frac{3}{2} w^2 + \frac{7}{2} w^3 + \frac{87}{8} w^4 + \frac{1581}{40} w^5 + \frac{12531}{80} w^6 + \cdots$$

which is the EGF of the sequence

$$1, 3, 21, 261, 4743, 112779, 3297213, 114135093, 4561146639, \\ 206614134747, 10461295740789, \cdots$$

We are now in a position to define the inverse transform, which is such that

$$a_n = \sum_{k=0}^n \left\|\begin{matrix} n \\ k \end{matrix}\right\|_q b_k.$$

Here we again have

$$\left\|\begin{matrix} n \\ k \end{matrix}\right\|_1 = \left\{\begin{matrix} n \\ k \end{matrix}\right\}.$$

These numbers also represent set partitions and we obtain

$$\left\|\begin{matrix} n \\ k \end{matrix}\right\|_3 = n! [w^n] \frac{G(w)^k}{k!}.$$

The following table displays the values for up to $n = 10$. For q other than 3 we just replace $\tau_3(m)$ by $\tau_q(m)$ in the definition of $f(t)$ for the inversion computation.

1									
3	1								
21	9	1							
261	111	18	1						
4743	1935	345	30	1					
112779	44613	8100	825	45	1				
3297213	1280097	232533	25200	1680	63	1			
114135093	43813143	7946694	885213	64890	3066	84	1		
4561146639	1738235295	315065565	35624610	2745603	146286	5166	108	1	
206614134747	78344599869	14202313200	1621932525	128802555	7351911	298620	8190	135	1

For example with the source sequence $a_n = 2^n$ we get the transform

$$1, 2, -2, -16, 4, -184, -1688, 1088, -13616, \\ 700064, 12016096, \cdots$$

and on substituting this into the inverse transform we indeed obtain

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \cdots$$

A key question

As we have seen earlier the numbers $\left|{n \atop k}\right|$ have a simple combinatorial interpretation as set partitions into q-tori. The question now becomes, what is the combinatorial meaning of $\left\|{n \atop k}\right\|$? As observed, it represents a set partition of some kind by construction. We ask into what type of object. These are enumerated by the corresponding $G_q(w)$. Also of interest are the row sums of this sequence which have bivariate EGF

$$\exp(u \times G_q(w))$$

so that

$$\left\|{n \atop k}\right\|_q = n![w^n][u^k]\exp(u \times G_q(w)).$$

This will produce the sequence starting at $n = 0$

$$1, 1, 4, 31, 391, 7054, 166363, 4836787, \\ 166848184, 6652969273, \cdots$$

Addendum

It is important to verify that the inversion will indeed produce the EGF of the Stirling set numbers when $q = 1$. We get from the cited formula

$$\frac{1}{n} \operatorname*{res}_t \frac{1}{t^n} \left(\frac{t}{f(t)}\right)^n = \frac{1}{n} \operatorname*{res}_t \frac{1}{\log^n(1+t)}.$$

Put $\log(1+t) = u$ so that $t = \exp(u) - 1$ to get

$$\frac{1}{n} \operatorname*{res}_u \frac{1}{u^n} \exp(u) = \frac{1}{n}[u^{n-1}]\exp(u) = \frac{1}{n}\frac{1}{(n-1)!} = \frac{1}{n!}$$

and the sanity check goes through. There is exactly one set on n labels.